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INCLUSION OF AERODYNAMIC DRAG IN VINTI'S SATELLITE THEORY

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ABSTRACT

In order to achieve separability in the Vinti theory of earth satellite motion when a non-conservative force such as air drag is considered, it is presently necessary to introduce a set of variational equations for the orbital elements, expressed as functions of the orbital elements and the tangential, radial, and normal components of the non-conservative forces acting on the system. The presence of a velocity dependent potential no longer allows the Hamiltonian to be the total energy, and furthermore, this term is cubic in the velocity. In this approach, the Hamiltonian is preserved in form, and remains as the total energy but the initial conditions and hence the Jacobi constants of the motion are changed with time through the variational equations. Hence separability is achieved at each point in time corresponding to continuously changing generalized coordinates in phase space.

Some initial results of application of this program to the heavy air drag satellite San Marco 2, show excellent improvement in the accuracy of the computed orbits.

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INTRODUCTION

In this paper we shall treat the variation of Izsak elements caused by disturbing forces acting on an orbit. These elements, an intrinsic part of the Vinti theory, are obtained as functions of time and then by a process of re-initialization, the Vinti equations of motion are solved for the position and velocity of the satellite. We should note that the air resistance of a body has in general six components, three being forces and three being moments of forces, which tend to make the motion of an asymmetric body very complex. However, for the case of a non-rotating sphere, the resistance can be reduced to a single component directed oppositely to the velocity of the sphere. In this treatment we shall introduce semi-empirically determined components of drag force which are functions of the eccentric anomaly. With use of these, the differential equations for the variation with time of the Izsak elements can be solved along with the Vinti equations of motion.

I. DRAG RATE EQUATIONS

Here we present equations for the time rate of change of the Izsak orbital elements of the Vinti satellite theory (Reference 2). Since they can be shown to result from a consideration of conservation of energy and angular momentum, these equations hold for all perturbing forces. In our treatment, we shall assume

that these equations represent instantaneous departures due to aerodynamic drag or perturbing forces on a Vinti orbit. These departures in the Vinti-Izsak orbital elements are then integrated as functions of eccentric anomaly for the instant of time under consideration.

The differential equations representing variations of orbital elements are:

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial \bar{U}}{\partial M},$$

$$\frac{de}{dt} = \frac{1-e^2}{na^2 e}\frac{\partial \bar{U}}{\partial M} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial \bar{U}}{\partial \omega}$$

$$\frac{d\omega}{dt} = \frac{-\cos i}{na^2\sqrt{1-e^2}\,\sin i}\left(\frac{\partial \bar{U}}{\partial i}\right) + \frac{\sqrt{1-e^2}}{na^2 e}\left(\frac{\partial \bar{U}}{\partial e}\right)$$

$$\frac{di}{dt} = \frac{\cos i}{na^2\sqrt{1-e^2}\,\sin i}\left(\frac{\partial \bar{U}}{\partial \omega}\right)$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1-e^2}\,\sin i}\left(\frac{\partial \bar{U}}{\partial i}\right)$$

$$\frac{dM}{dt} = n - \frac{1-e^2}{na^2 e}\left(\frac{\partial \bar{U}}{\partial e}\right) - \frac{2}{na}\frac{\partial \bar{U}}{\partial a} \qquad (1)$$

where n is related to a by $n^2 a^3 = GM = \mu$. G is the gravitational constant, and M is the mass of the Earth. $\bar{U}$, is any disturbing function due to the oblateness of the Earth. In order to get these equations in terms of a perturbing force F,

we assume that this perturbing force may be described by the gradient of a perturbing potential. We now resolve the perturbing force F into the following components:

R, in the direction of the position vector from the force center to the satellite.

T, is perpendicular to R, lies in the orbital plane, and is positive in the direction of motion.

W, is mutually perpendicular to R and T and completes a right handed set of component directions.

The partial derivatives appearing in Equations (1) are given by:

$$\frac{\partial \bar{U}}{\partial a} = R\left(\frac{r}{a}\right)$$

$$\frac{\partial \bar{U}}{\partial e} = -R\,a\cos v + T\left(\frac{2 + e\cos v}{1 - e^2}\right) r\sin v$$

$$\frac{\partial \bar{U}}{\partial a} = r\,W\sin\psi$$

$$\frac{\partial \bar{U}}{\partial \Omega} = T\,r\cos i - W(r\cos\psi\sin i)$$

$$\frac{\partial \bar{U}}{\partial \omega} = r\,T$$

$$\frac{\partial \bar{U}}{\partial M} = R\left(\frac{a\,e}{\sqrt{1 - e^2}}\right)\sin v + T\left(\frac{a^2\sqrt{1 - e^2}}{r}\right) \qquad (2)$$

where ψ is the argument of latitude. Substituting (2) into (1),

$$\frac{da}{dt} = 2\left(\frac{a^3}{\mu}\right)^{1/2} \frac{1}{\sqrt{1-e^2}} \left[Re \sin v + T(1 + e \cos v)\right]$$

$$\frac{de}{dt} = \left(\frac{a}{\mu}\right)^{1/2} \sqrt{1-e^2} \left[R \sin v + T(\cos v + \cos E)\right]$$

$$\frac{di}{dt} = W\left(\frac{r \cos\psi}{na^2\sqrt{1-e^2}}\right)$$

$$\frac{d\omega}{dt} = -R\left(\frac{\sqrt{1-e^2}\cos v}{n a e}\right) + T\left(\frac{\sqrt{1-e^2}\left(1+\frac{r}{p}\right)\sin v}{n a e}\right) - W\left(\frac{r \cos i \sin\psi}{na^2\sqrt{1-e^2}\ \sin i}\right)$$

$$\frac{d\Omega}{dt} = W\left(\frac{r \sin\psi}{na^2\sqrt{1-e^2}\ \sin i}\right)$$

$$\frac{dM}{dt} = n + R\left(\frac{\left(1-e^2\right)}{nae}\cos v - \frac{2r}{na^2}\right) - T\left(\frac{\left(1-e^2\right)\left(1+\frac{r}{p}\right)\sin v}{nae}\right). \tag{3}$$

The Vinti parameter related to the orbit inclination is

$$S = \sin^2 i \tag{4}$$

From this, we have

$$\frac{dS}{dt} = 2 \sin i \cos i \frac{di}{dt},$$

or,

$$\frac{dS}{dt} = 2\sqrt{S}\sqrt{1-S}\,\frac{di}{dt} . \tag{5}$$

The variation of the inclination can then be written as

$$\frac{dS}{dt} = \frac{2\sqrt{S(1-S)}\ r W \cos\psi}{\sqrt{\mu a}\ \sqrt{1-e^2}} . \tag{6}$$

Let $\Omega = \beta_3$, $\omega = \beta_2$, and using

$$p = a\left(1-e^2\right) ,$$

$$n^2 a^3 = \mu ,$$

$$\sin i = \sqrt{S} ,$$

we have,

$$\frac{d\beta_2}{dt} = -\left\{\frac{\sqrt{S}\left[aR\left(1-e^2\right)\cos v - r(2+e\cos v)\,T\sin v\right] + r\,W e \sin\psi\sqrt{1-S}}{(\mu a)^{1/2}\, e\sqrt{S}\sqrt{1-e^2}}\right\}$$

and

$$\frac{d\beta_3}{dt} = \frac{r\,W\sin\psi}{(\mu a)^{1/2}\sqrt{S}\sqrt{1-e^2}} \tag{7}$$

The mean anomaly is related to the time of perigee passage β_1 by the expression,

$$M = n\left(t + \beta_1\right) \tag{8}$$

From this,

$$\frac{dM}{dt} = \frac{dn}{dt}\left(t + \beta_1\right) + n\,\frac{d\beta_1}{dt} + n$$

or,

$$\frac{d\beta_1}{dt} = \frac{1}{n}\left[\frac{dM}{dt} - \left(t + \beta_1\right)\frac{dn}{dt} - n\right] \tag{9}$$

and

$$\frac{dn}{dt} = -\frac{3}{2}\,\mu^{1/2}\,a^{-5/2}\,\frac{da}{dt} \tag{10}$$

together with dM/dt from (3), we find

$$\frac{d\beta_1}{dt} = -\frac{3}{2}\,\beta_1\,a^{-1}\,\frac{da}{dt} + \mu(-2\alpha_1)^{-3/2}\left\{\frac{3}{2}\,t\,\mu^{1/2}\,a^{-5/2}\,\frac{da}{dt} - \frac{2r\,R}{(\mu a)^{1/2}}\right.$$

$$\left. + \frac{\left(1 - e^2\right)}{e}\left(\frac{a}{\mu}\right)^{1/2}\left[R\cos v - T\left(1 + \frac{r}{a\left(1 - e^2\right)}\right)\right]\right\} \cdot \tag{11}$$

Summarizing our equations for the time rate of change of orbital elements, we have:

$$\frac{da}{dt} = 2\left(\frac{a^3}{\mu}\right)^{1/2}\frac{1}{\sqrt{1 - e^2}}\left[Re\,\sin v + T(1 + e\cos v)\right]$$

$$\frac{de}{dt} = \left(\frac{a}{\mu}\right)^{1/2} \sqrt{1-e^2}\left[R \sin v + T(\cos v + \cos E)\right]$$

$$\frac{dS}{dt} = \frac{2\sqrt{S(1-S)}\; rW\cos\psi}{(\mu a)^{1/2}\sqrt{1-e^2}}$$

$$\frac{d\beta_3}{dt} = \frac{rW\sin\psi}{(\mu a)^{1/2}\sqrt{S}\sqrt{1-e^2}}$$

$$\frac{d\beta_2}{dt} = -\left\{\frac{\sqrt{S}\left[aR(1-e^2)\cos v - r(2+e\cos v)T\sin v\right] + reW\sin\psi\sqrt{1-S}}{(\mu a)^{1/2}\sqrt{S}\,e\sqrt{1-e^2}}\right\}$$

$$\frac{d\beta_1}{dt} = -\frac{3}{2}\beta_1 a^{-1}\frac{da}{dt} + \mu\left(-2\alpha_1\right)^{-3/2}\left\{\frac{3}{2} t\mu^{1/2} a^{-5/2}\frac{da}{dt} - \frac{2rR}{(\mu a)^{1/2}}\right.$$

$$\left. + \frac{\left(1-e^2\right)}{e}\left(\frac{a}{\mu}\right)^{1/2}\left[R\cos v - T\left(1+\frac{r}{a\left(1-e^2\right)}\right)\right]\right\}. \qquad (12)$$

II. SOLUTION OF DRAG-RATE EQUATIONS

Let us now substitute for R, T, and W in terms of the eccentric anomaly as given in Reference 1, page 165:

$$R = -\frac{1}{2} C_D \frac{A}{m} ae\rho V \sin E \frac{dE}{dt}$$

$$T = -\frac{1}{2} C_D \frac{A}{m}(1-e^2)^{1/2} a\rho V\left[1-d\frac{(1-e\cos E)^2}{(1-e^2)}\right]\frac{dE}{dt} \qquad (1)$$

$$W = -\frac{1}{2} C_D \frac{A}{m} a\rho\omega_s V\mu^{-1/2} a^{3/2}$$

$$\times (1 - e\cos E)^2 \sqrt{s} \cos\psi \frac{dE}{dt}$$

where V is the velocity of the satellite defined by the equation

$$V = \left(\frac{\mu}{a}\right)^{1/2} (1 + e\cos E)^{-1/2} (1 - e\cos E)^{-1/2} \Big[(1 + e\cos E)$$

$$- d(1 + e\cos E)\Big] \quad (2)$$

where

$$d = \omega_s \mu^{-1/2} a^{3/2} (1 - e^2)^{1/2} (1 - S)^{1/2} \quad (3)$$

where

ω_s = angular velocity of rotation of earth

ρ = atmospheric density

A = projected area of satellite

m = mass of satellite

C_D = drag parameter (usually around 2.2)

$$\cos v = \frac{\cos E - e}{1 - e\cos E}$$

$$\sin v = \frac{(1-e^2)^{1/2}\sin E}{1-e\cos E}$$

$$r = a(1-e\cos E)\ . \tag{4}$$

Now substituting Equations (2) and (3) into Equations (1), and inserting these results with Equations (4) into Equations (12) of Section I, we get

$$\frac{da}{dt} = -C_D\frac{A}{m}\rho a^2\left\{(1-e\cos E)^{-1/2}(1+e\cos E)^{-1/2}\left[(1+e\cos E) - d(1-e\cos E)\right]^2\right\}\frac{dE}{dt}$$

$$\frac{de}{dt} = -\frac{1}{2}C_D\frac{A}{m}\rho a(1-e^2)\left\{\left[(1+e\cos E)^{-1/2}(1-e\cos E)^{-3/2}\left[(1+e\cos E) - d(1-e\cos E)\right]\right] + \left[e\sin^2 E + (1-e^2)^{-1}\left[(1-e^2) - d(1-e\cos E)^2\right]\left[(\cos E-e)+\cos E(1-e\cos E)\right]\right]\right\}\frac{dE}{dt}$$

$$\frac{dS}{dt} = -C_D \frac{A}{m} \rho \omega_s \sqrt{S(1-S)}\ \mu^{-1/2} a^{5/2} (1-e^2)^{-1/2}$$

$$\times \left\{\cos^2 \psi (1 - e\cos E)^{5/2} (1 + e\cos E)^{-1/2} \left[(1 + e\cos E) - d(1 - e\cos E)\right]\right\} \frac{dE}{dt}$$

$$\frac{d\beta_2}{dt} = -\frac{1}{2} C_D \frac{A}{m} \rho a e^{-1} (1-e^2)^{-1/2} \Big\{ e \sin\psi \cos\psi (1-S)^{1/2} \omega_s \mu^{-3/2} a^{3/2}$$

$$\times (1 + e\cos E)^{-1/2} (1 - e\cos E)^{5/2} \big[(1 + e\cos E)$$

$$- d(1 - e\cos E)\big] + (1 - e\cos E)^{-3/2} (1 + e\cos E)^{-1/2} \big[(1 + e\cos E)$$

$$- d(1 - e\cos E)\big] \big[2 \sin E (1 - d) + d e^2 \sin E (1 + e^2 \cos^2 E) + d e \sin E \cos E$$

$$- 4e^2 \sin E (1 + d\cos^2 E) + 2e^4 \sin E (5 - 2e^2 + e^2 \cos^2 E)\big]\Big\} \frac{dE}{dt}$$

$$\frac{d\beta_3}{dt} = \frac{1}{2} C_D \frac{A}{m} \rho \omega_s \mu^{-1/2} a^{5/2} (1-e^2)^{-1/2}$$

$$\times \Big\{ \sin\psi \cos\psi (1 - e\cos E)^{5/2} (1 + e\cos E)^{-1/2} \big[(1 + e\cos E)$$

$$- d(1 - e\cos E)\big]\Big\} \frac{dE}{dt}$$

$$\frac{d\beta_1}{dt} = -\mu(-2\alpha_1)^{-3/2}\left\{\frac{3}{2}\, t\, \mu^{1/2} a^{-5/2} \frac{da}{dt}\right.$$

$$+ (1-e^2)^{1/2}\left[\frac{d\beta_2}{dt} + \sqrt{1-S}\,\frac{d\beta_3}{dt}\right]$$

$$+ C_D \frac{A}{m} \rho a e \left[\sin E (1 - e\cos E)^{1/2} (1 + e\cos E)^{-1/2} \left[(1 + e\cos E)\right.\right.$$

$$\left.\left.\left. - d(1 - e\cos E)\right]\right] \frac{dE}{dt}\right\} - \frac{3}{2}\frac{1}{a}\left(t_0 - \beta_1\right)\frac{da}{dt} \quad . \qquad (5)$$

We now integrate Equations (5) from time t_1 to time t_2 and indicate the eccentric anomaly at time t_1 by E_1 and the eccentric anomaly at time t_2 by E_2. We have the following equations:

$$\Delta a = -C_D \frac{A}{m} a^2 \int_{E_1}^{E_2} \rho \left\{(1 - e\cos E)^{-1/2} (1 + e\cos E)^{-1/2} \left[(1 + e\cos E)\right.\right.$$

$$\left.\left. - d(1 - e\cos E)\right]^2\right\} dE$$

$$\Delta e = -\frac{1}{2} C_D \frac{A}{m} (1 - e^2) \int_{E_1}^{E_2} \rho \left\{\left[(1 + e\cos E)^{-1/2} (1 - e\cos E)^{-3/2} \left[(1 + e\cos E)\right.\right.\right.$$

$$\left.\left. - d(1 - e\cos E)\right]\right] \left[e \sin^2 E + (1 - e^2)^{-1} \left[(1 - e^2)\right.\right.$$

$$\left.\left.\left. - d(1 - e\cos E)^2\right] \left[(\cos E - e) + \cos E(1 - e\cos E)\right]\right]\right\} dE$$

$$\Delta S = - C_D \frac{A}{m} \omega_s S \sqrt{1 - S} \mu^{-1/2} a^{5/2} (1 - e^2)^{-1/2}$$

$$\times \int_{E_1}^{E_2} \rho \cos \psi \{(1 - \cos E)^{5/2} (1 + e \cos E)^{-1/2} [(1 + e \cos E) - d (1 - e \cos E)]\} dE$$

$$\Delta \beta_2 = -\frac{1}{2} C_D \frac{A}{m} a e^{-1} (1 - e^2)^{-1/2} \int_{E_1}^{E_2} \rho \{e \sin \psi \cos \psi (1 - S)^{1/2} \omega_s \mu^{-3/2} a^{3/2}$$

$$\times (1 + e \cos E)^{-1/2} (1 - e \cos E)^{5/2} [(1 + e \cos E)$$

$$- d (1 - e \cos E)] + (1 - e \cos E)^{-3/2} (1 + e \cos E)^{-1/2} [(1 + e \cos E)$$

$$- d (1 - e \cos E)] [2 \sin E (1 - d) - 4 a^2 \sin E (1 + d \cos^2 E) + 2 e^4 \sin E$$

$$+ d e^2 \sin E (1 + e^2 \cos^2 E) + d e \sin E \cos E (5 - 2 e^2 + e^2 \cos^2 E)]\} dE$$

$$\Delta \beta_3 = \frac{1}{2} C_D \frac{A}{m} \omega_s \mu^{-1/2} a^{5/2} (1 - e^2)^{-1/2}$$

$$\times \int_{E_1}^{E_2} \rho \sin \psi \cos \psi (1 - e \cos E)^{5/2} (1 + e \cos E)^{-1/2} [(1 + e \cos E)$$

$$- d (1 - e \cos E)] dE$$

$$\Delta \beta_1 = - \mu (- 2 \alpha_1)^{-3/2} \left\{ \frac{3}{2} t \mu^{1/2} a^{-5/2} \Delta a \right.$$

$$+ (1 - e^2)^{1/2} [\Delta\beta_2 + (2 - \sqrt{1 - S}) \Delta \beta_3]$$

$$+ C_D \frac{A}{m} a e . \int_{E_1}^{E_2} \rho \sin E (1 - e \cos E)^{1/2} (1 + e \cos E)^{-1/2} [(1 + e \cos E)$$

$$\left. - d (1 - e \cos E)] d E \right\} - \frac{3}{2} \frac{1}{a} (t_0 - \beta_1) \Delta a \tag{6}$$

Since ρ is a function of the eccentric anomaly also, then equations (6) are of the form

$$\int_{E_1}^{E_2} \rho (E) d G (E),$$

which can be integrated by parts. Following this, we can then write the solutions of equations (6) as,

$$\int_{E_1}^{E_2} \rho (E) d G (E) = \rho (E) G (E) \Big|_{E_1}^{E_2} - \int_{E_1}^{E_2} G (E) d \rho (E) \tag{6.a}$$

In order to find the functions G(E), we expand equations (6), keeping $\rho(E)$ only as a factor. From this we obtain the terms to be integrated,

$$\int_{E_1}^{E_2} (1 - e \cos E)^{-1/2} (1 + e \cos E)^{5/2} d E \tag{7}$$

$$\int_{E_1}^{E_2} (1 - e\cos E)^{-1/2} (1 + e\cos E)^{1/2}\, dE \tag{8}$$

$$\int_{E_1}^{E_2} (1 - e\cos E)^{1/2} (1 + e\cos E)^{-1/2}\, dE \tag{9}$$

$$\int_{E_1}^{E_2} (1 - e\cos E)^{1/2} (1 + e\cos E)^{3/2}\, dE \tag{10}$$

$$\int_{E_1}^{E_2} (1 - e\cos E)^{3/2} (1 + e\cos E)^{1/2}\, dE \tag{11}$$

$$\int_{E_1}^{E_2} (1 - e\cos E)^{5/2} (1 + e\cos E)^{-1/2}\, dE \tag{12}$$

$$\int_{E_1}^{E_2} (1 - e\cos E)^{5/2} (1 + e\cos E)^{1/2}\, dE \tag{13}$$

$$\int_{E_1}^{E_2} (1 - e\cos E)^{7/2} (1 + e\cos E)^{-1/2}\, dE \tag{14}$$

$$\int_{E_1}^{E_2} \cos E(1 - e\cos E)^{5/2} (1 + e\cos E)^{1/2}\, dE \tag{15}$$

$$\int_{E_1}^{E_2} \sin E(1 - e\cos E)^{-5/2} (1 + e\cos E)^{-1/2}\, dE \tag{16}$$

$$\int_{E_1}^{E_2} \sin E(1 - e\cos E)^{-3/2} (1 + e\cos E)^{1/2}\, dE \tag{17}$$

$$\int_{E_1}^{E_2} \sin E(1 - e\cos E)^{-1/2} (1 + e\cos E)^{1/2}\, dE \tag{18}$$

$$\int_{E_1}^{E_2} \sin E(1 - e\cos E)^{3/2} (1 + e\cos E)^{-1/2}\, dE \tag{19}$$

$$\int_{E_1}^{E_2} \sin E\cos^2 E(1 - e\cos E)^{-5/2} (1 + e\cos E)^{-1/2}\, dE \tag{20}$$

$$\int_{E_1}^{E_2} \sin E\cos^2 E(1 - e\cos E)^{-3/2} (1 + e\cos E)^{1/2}\, dE \tag{21}$$

$$\int_{E_1}^{E_2} \sin E\cos E(1 - e\cos E)^{-5/2} (1 + e\cos E)^{-1/2}\, dE \tag{22}$$

$$\int_{E_1}^{E_2} \sin E\cos E(1 - e\cos E)^{-3/2} (1 + e\cos E)^{1/2}\, dE \tag{23}$$

$$\int_{E_1}^{E_2} \sin E \cos^3 E(1 - e \cos E)^{-5/2} (1 + e \cos E)^{-1/2} \, dE \tag{24}$$

$$\int_{E_1}^{E_2} \sin E \cos^3 E(1 - e \cos E)^{-3/2} (1 + e \cos E)^{1/2} \, dE \tag{25}$$

$$\int_{E_1}^{E_2} \cos E(1 - e \cos E)^{7/2} (1 + e \cos E)^{-1/2} \, dE \tag{26}$$

$$\int_{E_1}^{E_2} \cos^2 E(1 - e^2 \cos^2 E)^{1/2} \, dE \tag{27}$$

$$\int_{E_1}^{E_2} \cos E(1 - e^2 \cos^2 E)^{1/2} \, dE \tag{28}$$

$$\int_{E_1}^{E_2} \cos E \sin E(1 - e^2 \cos^2 E)^{1/2} \, dE \tag{29}$$

$$\int_{E_1}^{E_2} \sin E(1 - e^2 \cos^2 E)^{1/2} \, dE \tag{30}$$

$$\int_{E_1}^{E_2} \sin^2 E(1 - e^2 \cos^2 E)^{1/2} \, dE \tag{31}$$

$$\int_{E_1}^{E_2} \cos^2 E(1 - e\cos E)^{3/2} (1 + e\cos E)^{-1/2}\, dE \tag{32}$$

$$\int_{E_1}^{E_2} \cos E(1 - e\cos E)^{3/2} (1 + e\cos E)^{-1/2}\, dE \tag{33}$$

$$\int_{E_1}^{E_2} \cos E \sin E(1 - e\cos E)^{3/2} (1 + e\cos E)^{-1/2}\, dE \tag{34}$$

$$\int_{E_1}^{E_2} \sin E(1 - e\cos E)^{3/2} (1 + \cos E)^{-1/2}\, dE \tag{35}$$

$$\int_{E_1}^{E_2} \sin^2 E(1 - e\cos E)^{3/2} (1 + e\cos E)^{-1/2}\, dE \tag{36}$$

Some of the integrals (Equations 7-36) may be expanded into two or more integrals. Equation (7) may be used as an example.

$$\int_{E_1}^{E_2} (1 - e\cos E)^{-1/2} (1 + e\cos E)^{5/2}\, dE$$

$$= \int_{E_1}^{E_2} (1 - e\cos E)^{-1/2} (1 + e\cos E)^{4/2} (1 + e\cos E)^{1/2}\, dE$$

$$= \int_{E_1}^{E_2} (1 - e\cos E)^{-1/2} (1 + e\cos E)^{1/2} (1 + 2e\cos E + e^2\cos^2 E)\, dE$$

$$= \int_{E_1}^{E_2} (1 - e\cos E)^{-1/2} (1 + e\cos E)^{1/2}\, dE$$

$$+ 2\int_{E_1}^{E_2} e\cos E(1 - e\cos E)^{-1/2} (1 + e\cos E)^{1/2}\, dE$$

$$+ \int_{E_1}^{E_2} e^2\cos^2 E(1 - e\cos E)^{-1/2} (1 + e\cos E)^{1/2}\, dE. \quad (37)$$

Using similar methods on Equations 8-36, we have the result

$$\int_{E_1}^{E_2} (1 - e\cos E)^{-1/2} (1 + e\cos E)^{5/2}\, dE$$

$$= \int_{E_1}^{E_2} (1 - e\cos E)^{-1/2} (1 + e\cos E)^{-1/2}\, dE$$

$$+ 2\int_{E_1}^{E_2} e\cos E (1 - e\cos E)^{-1/2} (1 + e\cos E)^{1/2}\, dE \quad .$$

$$+ \int_{E_1}^{E_2} e^2 \cos^2 E (1 - e\cos E)^{-1/2} (1 + e\cos E)^{1/2}\, dE \quad (37)$$

$$\int_{E_1}^{E_2} (1 - e\cos E)^{1/2} (1 + e\cos E)^{3/2}\, dE$$

$$= \int_{E_1}^{E_2} (1 - e^2\cos^2 E)^{1/2}\, dE + \int_{E_1}^{E_2} e\cos E (1 - e^2\cos^2 E)\, dE \quad (38)$$

$$\int_{E_1}^{E_2} (1 - e\cos E)^{3/2} (1 + e\cos E)^{1/2}\, dE$$

$$= \int_{E_1}^{E_2} (1 - e^2 \cos^2 E)^{1/2}\, dE$$

$$- \int_{E_1}^{E_2} e\cos E(1 - e^2 \cos^2 E)^{1/2}\, dE \quad (39)$$

$$\int_{E_1}^{E_2} (1 - e\cos E)^{5/2} (1 + e\cos E)^{-1/2}\, dE$$

$$= \int_{E_1}^{E_2} (1 - e\cos E)^{1/2} (1 + e\cos E)^{-1/2}\, dE$$

$$- 2 \int_{E_1}^{E_2} e\cos E(1 - e\cos E)^{1/2} (1 + e\cos E)^{-1/2}\, dE$$

$$+ \int_{E_1}^{E_2} e^2 \cos^2 E(1 - e\cos E)^{1/2} (1 + e\cos E)^{-1/2}\, dE \quad (40)$$

$$\int_{E_1}^{E_2} (1 - e \cos E)^{5/2} (1 + e \cos E)^{1/2} \, dE$$

$$= \int_{E_1}^{E_2} (1 - e^2 \cos^2 E)^{1/2} \, dE - 2 \int_{E_1}^{E_2} e \cos E (1 - e^2 \cos^2 E)^{1/2} \, dE$$

$$+ \int_{E_1}^{E_2} e^2 \cos^2 E (1 - e^2 \cos^2 E)^{1/2} \, dE \quad (41)$$

$$\int_{E_1}^{E_2} (1 - e \cos E)^{7/2} (1 + e \cos E)^{-1/2} \, dE$$

$$= \int_{E_1}^{E_2} (1 - e \cos E)^{1/2} (1 + e \cos E)^{-1/2} \, dE$$

$$- 3 \int_{E_1}^{E_2} e \cos E (1 - e \cos E)^{1/2} (1 + e \cos E)^{-1/2} \, dE$$

$$+ 3 \int_{E_1}^{E_2} e^2 \cos^2 E (1 - e \cos E)^{1/2} (1 + e \cos E)^{-1/2} \, dE$$

$$- \int_{E_1}^{E_2} e^3 \cos^3 E (1 - e \cos E)^{1/2} (1 + e \cos E)^{-1/2} \, dE \quad (42)$$

$$\int_{E_1}^{E_2} \cos E(1 - e\cos E)^{5/2} (1 + e\cos E)^{1/2}\, dE$$

$$= \frac{1}{e}\left[\int_{E_1}^{E_2} e\cos E(1 - e^2\cos^2 E)^{1/2}\, dE\right.$$

$$- 2\int_{E_1}^{E_2} e^2\cos^2 E(1 - e^2\cos^2 E)^{1/2}\, dE$$

$$\left. + \int_{E_1}^{E_2} e^3\cos^3 E(1 - e^2\cos^2 E)^{1/2}\, dE\right] \quad (43)$$

$$\int_{E_1}^{E_2} \sin E(1 - e\cos E)^{3/2} (1 + e\cos E)^{-1/2}\, dE$$

$$= \int_{E_1}^{E_2} \sin E(1 - e\cos E)^{1/2} (1 + e\cos E)^{-1/2}\, dE$$

$$- \int_{E_1}^{E_2} e\sin E\cos E(1 - e\cos E)^{1/2} (1 + e\cos E)^{-1/2}\, dE \quad (44)$$

$$\int_{E_1}^{E_2} \sin E \cos^2 E(1 - e\cos E)^{-5/2} (1 + e\cos E)^{-1/2}\, dE$$

$$= \int_{E_1}^{E_2} \sin E(1 - e\cos E)^{-2} (1 - e^2\cos^2 E)^{-1/2}\, dE$$

$$- \int_{E_1}^{E_2} \sin^3 E(1 - e\cos E)^{-2} (1 - e^2\cos^2 E)^{-1/2}\, dE \quad (45)$$

$$\int_{E_1}^{E_2} \sin E \cos^2 E(1 - e\cos E)^{-3/2} (1 + e\cos E)^{1/2}\, dE$$

$$= \int_{E_1}^{E_2} \sin E(1 - e\cos E)^{-3/2} (1 + e\cos E)^{1/2}\, dE$$

$$- \int_{E_1}^{E_2} \sin^3 E(1 - e\cos E)^{-3/2} (1 + e\cos E)^{1/2}\, dE \quad (46)$$

$$\int_{E_1}^{E_2} \sin E \cos^3 E(1 - e\cos E)^{-5/2} (1 + e\cos E)^{-1/2}\, dE$$

$$= \int_{E_1}^{E_2} \sin E \cos E(1 - e\cos E)^{-2} (1 - e^2\cos^2 E)^{-1/2}\, dE$$

$$- \int_{E_1}^{E_2} \sin^3 E \cos E(1 - e\cos E)^{-2} (1 - e^2\cos^2 E)^{-1/2}\, dE \quad (47)$$

$$\int_{E_1}^{E_2} \sin E \cos^3 E(1 - e\cos E)^{-3/2} (1 + e\cos E)^{1/2}\, dE$$

$$= \int_{E_1}^{E_2} \sin E \cos E(1 - e\cos E)^{-3/2} (1 + e\cos E)^{1/2}\, dE$$

$$- \int_{E_1}^{E_2} \sin^3 E \cos E(1 - e\cos E)^{-3/2} (1 + e\cos E)^{1/2}\, dE \quad (48)$$

Below is the series solution for the incomplete elliptic integrals of the first and second kind.

$$F(\alpha, \phi) = \phi\left(1 + \frac{1}{4} k^2 + \frac{9}{64} k^4 + \frac{25}{256} k^6 + \frac{(49)(25)}{(64)(256)} k^8\right)$$

$$- \frac{1}{4} k^2 \sin\phi\cos\phi\left(1 + \frac{9}{16} k^2 + \frac{25}{64} k^4 + \frac{49}{256}\,\frac{25}{64} k^6\right)$$

$$- \frac{1}{32} k^4 \sin^3\phi\cos\phi\left(3 + \frac{25}{12} k^2 + \frac{1225}{12} \cdot \frac{1}{64} k^4\right)$$

$$- \frac{5}{96} k^6 \sin\phi\cos\phi\left(1 + \frac{49}{64} k^2\right) - \frac{35}{128} k^8 \sin^7\phi\cos\phi \quad (49)$$

$$E(\alpha, \phi) = \phi\left(1 - \frac{1}{4} k^2 - \frac{3}{64} k^4 - \frac{5}{256} k^6 - \frac{175}{16384} k^8\right)$$

$$+ \frac{1}{4} k^2 \sin\varphi \cos\phi \left(1 + \frac{3}{16} k^2 + \frac{5}{64} k^4 + \frac{175}{4096} k^6\right)$$

$$+ \frac{1}{32} k^4 \sin^3\phi \cos\phi \left(1 + \frac{5}{12} k^2 + \frac{175}{768} k^4\right)$$

$$+ \frac{1}{96} k^6 \sin^5\phi \cos\phi \left(1 + \frac{35}{64} k^2\right)$$

$$+ \frac{5}{1024} k^8 \sin^7\phi \cos\phi . \quad (50)$$

Here the values of k and ϕ depend upon the term that is being integrated and these values can be found in Reference 3. In addition, throughout the report we make use of the following two substitutions in Equations (49) and (50):

$$\phi = - E + \frac{\pi}{2}$$

and

$$k^2 = e^2$$

Examples of how the integration is performed is given in the appendix.

Here we give only the results of the integration.

$$\int_{E_1}^{E_2} (1 + e\cos E)^{1/2} (1 - e\cos E)^{-1/2}\, dE$$

$$= \left\{ \ln\left[(1 - e^2\cos^2 E)^{1/2} + e\sin E\right] + F(\alpha, \phi) \right\} \Bigg|_{E_1}^{E_2} \quad (51)$$

$$\int_{E_1}^{E_2} (1 - e\cos E)^{-1/2} (1 + e\cos E)^{5/2}\, dE$$

$$= \Bigg[\ln\left[(1 - e^2\cos^2 E)^{1/2} + e\sin E\right] + F(\alpha, \phi)$$

$$+ 2\Big\{ \ln\left[(1 - e^2\cos^2 E)^{1/2} + e\sin E\right]$$

$$+ E(\alpha, \phi) - F(\alpha, \phi) \Big\} + \frac{(1 - e^2)}{2} \ln\left[(1 - e^2\cos^2 E)^{1/2} + e\sin E\right]$$

$$- \frac{e\sin E(1 - e^2\cos^2 E)^{1/2}}{2} + E(\alpha, \phi) - F(\alpha, \phi) \Bigg] \Bigg|_{E_1}^{E_2}$$

$$= \Bigg\{ \frac{7 + e^2}{2} \ln\left[(1 - e^2\cos^2 E)^{1/2} + e\sin E\right]$$

$$+ 3\left[E(\alpha, \phi) - F(\alpha, \phi)\right] + F(\alpha, \phi) - \frac{e\sin E(1 - e^2\cos^2 E)^{1/2}}{2} \Bigg\} \Bigg|_{E_1}^{E_2} \quad (52)$$

$$\int_{E_1}^{E_2} (1 - e\cos E)^{1/2} (1 + e\cos E)^{-1/2}\, dE$$

$$= \left\{- 2F(\alpha, \phi) + E(\alpha, \phi)\right\}\Bigg|_{E_1}^{E_2} \tag{53}$$

$$\int_{E_1}^{E_2} (1 - e\cos E)^{1/2} (1 + e\cos E)^{3/2}\, dE = \Bigg[- E(\alpha, \phi) + \frac{1}{2}\Bigg\{e \sin E(1 - e^2 \cos^2 E)^{1/2}$$

$$+ (1 - e^2) \ln \left[\frac{e \sin E + (1 - e^2 \cos^2 E)^{1/2}}{(1 - e^2)^{1/2}}\right]\Bigg\}\Bigg]\Bigg|_{E_1}^{E_2} \tag{54}$$

$$\int_{E_1}^{E_2} (1 - e\cos E)^{3/2} (1 + e\cos E)^{1/2}\, dE = \Bigg[- E(\alpha, \phi)$$

$$- \frac{1}{2}\Bigg\{e \sin E(1 - e^2 \cos^2 E)^{1/2}$$

$$+ (1 - e^2) \ln \left[e \sin E + (1 - e^2 \cos^2 E)^{1/2}\right]\Bigg\}\Bigg]\Bigg|_{E_1}^{E_2} \tag{55}$$

$$\int_{E_1}^{E_2} (1 - e\cos E)^{5/2} (1 + e\cos E)^{-1/2}\, dE$$

$$= \left\{F(\alpha, \phi) + \frac{e \sin E(1 - e^2 \cos^2 E)}{2}\right.$$

$$- \frac{7 + e^2}{2} \ln \left[(1 - e^2 \cos^2 E)^{1/2} + e \sin E\right]$$

$$\left. + 3E(\alpha, \phi) - 3F(\alpha, \phi)\right\}\Bigg|_{E_1}^{E_2} . \quad (56)$$

$$\int_{E_1}^{E_2} (1 - e\cos E)^{7/2} (1 + e\cos E)^{-1/2}\, dE$$

$$= \left[F(\alpha, \phi) - 2(3 + e^2) \ln \left[(1 - e^2 \cos^2 E)^{1/2}\right.\right.$$

$$\left. + e \sin E\right] + 8E(\alpha, \phi) - 8F(\alpha, \phi)$$

$$+ 2e \sin E(1 - e^2 \cos^2 E)^{1/2} - \frac{1}{3} \frac{e^2 \cos^3 E(1 - e^2 \cos^2 E)^{1/2}}{\sin E}$$

$$- \frac{1}{3}\left\{E(\alpha, \phi) - (1 - e^2) F(\alpha, \phi) - e^2 \left[(1 + e^2) \frac{\sin\phi \cos\phi}{\sqrt{1 - e^2 \sin^2\phi}}\right.\right.$$

$$\left.\left.\left. + \frac{\cos\phi \sqrt{1 - e^2 \sin^2\phi}}{\sin\phi}\right]\right\}\right]\Bigg|_{E_1}^{E_2} . \quad (57)$$

$$\int_{E_1}^{E_2} \cos E(1 - e\cos E)^{5/2}\ (1 + e\cos E)^{1/2}\, dE$$

$$= \Bigg[\frac{5 + 3e^2}{8e}\Bigg\{e \sin E(1 - e^2\cos^2 E)^{1/2}$$

$$+ (1 - e^2)\ln\left[\frac{e\sin E + (1 - e^2\cos^2 E)^{1/2}}{(1 - e^2)^{1/2}}\right]\Bigg\}$$

$$+ \frac{2}{3}\ \frac{(1 - e^2\cos^2 E)^{1/2} + e\cos^3 E}{\sin E} + \frac{2}{3}\Bigg\{E(\alpha, \phi) - (1 - e^2)\,F(\alpha, \phi)$$

$$- e^2\left[\frac{(1 + e^2)\sin\phi\cos\phi}{\sqrt{1 - e^2\sin^2\phi}} + \frac{\cos\phi\sqrt{1 - e^2\sin^2\phi}}{\sin\phi}\right.$$

$$\left.\left. - \frac{1}{4}\, e\sin E(1 - e^2\cos^2 E)^{3/2}\right]\Bigg\}\Bigg]\Bigg|_{E_1}^{E_2} \qquad (58)$$

$$\int_{E_1}^{E_2} \sin E(1 - e\cos E)^{-5/2}\,(1 + e\cos E)^{-1/2}\, dE$$

$$= -\frac{1}{3e(1 - e^2\cos^2 E)^{3/2}}\ \left\{2 + 2e\cos E + e\cos E(1 - e^2\cos^2 E)\right\}\Bigg|_{E_1}^{E_2} \qquad (59)$$

$$\int_{E_1}^{E_2} \sin E(1 - e\cos E)^{-3/2}(1 + e\cos E)^{1/2}\, dE$$

$$= -\frac{1}{e(1 - e^2\cos^2 E)^{1/2}}\Big\{2(1 + e\cos E)$$

$$+ (1 - e^2\cos^2 E)^{1/2}\sin^{-1}\left[(1 - e^2\cos^2 E)^{1/2}\right]\Big\}\Bigg|_{E_1}^{E_2} \quad (60)$$

$$\int_{E_1}^{E_2} \sin E(1 - e\cos E)^{1/2}(1 + e\cos E)^{1/2}\, dE$$

$$= \frac{1}{2e}\Big\{\sin^{-1}\left[(1 - e^2\cos^2 E)^{1/2}\right.$$

$$\left. - e\cos E(1 - e^2\cos^2 E)^{1/2}\right]\Big\}\Bigg|_{E_1}^{E_2} \quad (61)$$

$$\int_{E_1}^{E_2} \sin E(1 - e\cos E)^{3/2}(1 + e\cos E)^{-1/2}\, dE$$

$$= \Bigg[\frac{1}{e}\Big\{\frac{3}{2}\sin^{-1}\left[(1 - e^2\cos^2 E)^{1/2}\right]$$

$$- 2(1 - e^2\cos^2 E)^{1/2} + \frac{1}{2}e\cos E(1 - e^2\cos^2 E)^{1/2}\Big\}\Bigg]\Bigg|_{E_1}^{E_2} \quad (62)$$

$$\int_{E_1}^{E_2} \sin E \cos^2 E (1 - e\cos E)^{-5/2} (1 + e\cos E)^{-1/2}\, dE$$

$$= -\frac{1}{3e(1 - e^2\cos^2 E)^{3/2}} \Bigg\{ 2 + 2e\cos E + e\cos E(1 - e^2\cos^2 E)$$

$$+ \frac{1}{e^2}\Big[-3(1 - e^2\cos^2 E)^{3/2} \sin^{-1}\big[(1 - e^2\cos^2 E)^{1/2}\big] + e\cos E(1 - e^2)(1 - e^2\cos^2 E)$$

$$+ (1 + e\cos E)\big[2(1 - e^2) - 6(1 - e^2\cos^2 E)\big]\Big]\Bigg\}\Bigg|_{E_1}^{E_2} \quad (63)$$

$$\int_{E_1}^{E_2} \sin E \cos^2 E (1 - e\cos E)^{-3/2} (1 + e\cos E)^{1/2}\, dE$$

$$= \Bigg[-\frac{1}{e(1 - e^2\cos^2 E)^{1/2}} \Big\{ 2(1 + e\cos E)$$

$$+ (1 - e^2\cos^2 E)^{1/2} \sin^{-1}\big[(1 - e^2\cos^2 E)^{1/2}\big] \Big\}$$

$$- \frac{1}{2e^3} \Big\{ (5 - 2e^2) \sin^{-1}\big[(1 - e^2\cos^2 E)^{1/2}\big]$$

$$+ 4(1 - e^2)(1 + e\cos E)(1 - e^2\cos^2 E)^{-1/2}$$

$$+ 4(1 - e^2\cos^2 E)^{1/2} + e\cos E(1 - e^2\cos^2 E)^{1/2} \Big\} \Bigg] \Bigg|_{E_1}^{E_2} \quad (64)$$

$$\int_{E_1}^{E_2} \sin E \cos E(1 - e \cos E)^{-5/2} (1 + e \cos E)^{-1/2} \, dE$$

$$= \frac{1}{3e^2 (1 - e^2 \cos^2 E)^{3/2}} \left[3(1 - e^2 \cos^2 E) - 2 - 2e^3 \cos^3 E\right] \Bigg|_{E_1}^{E_2} \quad (65)$$

$$\int_{E_1}^{E_2} \sin E \cos E(1 - e \cos E)^{-3/2} (1 + e \cos E)^{1/2} \, dE$$

$$= -\frac{1}{e^2 (1 - e^2 \cos^2 E)^{1/2}} \Big\{2(1 + e \cos E) + (1 - e^2 \cos^2 E)$$

$$+ 2(1 - e^2 \cos^2 E)^{1/2} \sin^{-1} \left[(1 - e^2 \cos^2 E)^{1/2}\right]\Big\} \Bigg|_{E_1}^{E_2} \quad (66)$$

$$\int_{E_1}^{E_2} \sin E \cos^3 E(1 - e \cos E)^{-5/2} (1 + e \cos E)^{-1/2} \, dE$$

$$= \frac{1}{3e^4 (1 - e^2 \cos^2 E)^{3/2}} \Big\{8 + 8e \cos E(1 - e^2 \cos^2 E)$$

$$- 3e^2 \cos^2 E(5 - e^2 \cos^2 E) - 2e^5 \sin^2 E \cos^3 E$$

$$+ 6(1 - e^2 \cos^2 E)^{3/2} \sin^{-1} \left[(1 - e^2 \cos^2 E)^{1/2}\right]\Big\} \Bigg|_{E_1}^{E_2} \quad (67)$$

$$\int_{E_1}^{E_2} \sin E \cos^3 E (1 - e\cos E)^{-3/2} (1 + e\cos E)^{1/2}\, dE$$

$$= -\left\{\frac{1}{e^2 (1 - e^2 \cos^2 E)^{1/2}} \Big[2(1 + e\cos E)\right.$$

$$+ (1 - e^2 \cos^2 E)$$

$$+ 2(1 - e^2 \cos^2 E)^{1/2} \sin^{-1} \left[(1 - e^2 \cos^2 E)^{1/2}\right]\Big]$$

$$+ \frac{1}{e^4} \Big[(1 - e^2 \cos^2 E)^{1/2} (3 - e^2 + e\cos E)$$

$$+ (3 - 2e^2) \sin^{-1} \left[(1 - e^2 \cos^2 E)^{1/2}\right.$$

$$- \frac{1}{3} (1 - e^2 \cos^2 E)^{3/2}$$

$$\left.\left. + 2(1 - e^2)(1 - e^2 \cos^2 E)^{-1/2} (1 + e\cos E)\right]\Big]\right\}\Bigg|_{E_1}^{E_2} \quad (68)$$

$$\int_{E_1}^{E_2} \cos E(1 - e\cos E)^{7/2} (1 + e\cos E)^{-1/2}$$

$$= \frac{1}{e}\left[\frac{e\sin E(1 - e^2\cos^2 E)^{3/2}}{4} + e\sin E(1 - e^2\cos^2 E)^{1/2}\left[\frac{3}{8}(1 - e^2) - 4\right]\right.$$

$$+ \left[\frac{3}{8}(1 - e^2)^2 - 4(1 - e^2) + 8\right] \ln\left[(1 - e^2\cos^2 E)^{1/2} + e\sin E\right]$$

$$- \frac{4}{3}\,\frac{(1 - e^2\cos^2 E)^{1/2} + e^2\cos^3 E}{\sin E} - 8E(\alpha, \phi) + 8F(\alpha, \phi)$$

$$- \frac{4}{3}\left\{E(\alpha, \phi) - (1 - e^2)F(\alpha, \phi)\right.$$

$$\left.\left.- e^2\left[(1 + e^2)\frac{\sin\phi\cos\phi}{(1 - e^2\sin^2\phi)^{1/2}} + \frac{\cos\phi(1 - e^2\sin^2\phi)^{1/2}}{\sin\phi}\right]\right\}\right]\Bigg|_{E_1}^{E_2} \tag{69}$$

$$\int_{E_1}^{E_2} \cos^2 E(1 - e^2\cos^2 E)^{1/2}\, dE = -\frac{1}{e^2}\left\{\frac{e^2\cos^3 E(1 - e^2\cos^2 E)^{1/2}}{3\sin E} + \frac{(1 - e^2)}{3}\,\Theta\right\}\Bigg|_{E_1}^{E_2}$$

(70)

$$\int_{E_1}^{E_2} \cos E(1 - e^2 \cos^2 E)^{1/2}\, dE = \frac{1}{2e}\left\{e \sin E(1 - e^2 \cos^2 E)^{1/2}\right.$$

$$\left. + (1 - e^2) \ln\left[e \sin E + (1 - e^2 \cos^2 E)^{1/2}\right]\right\}\Bigg|_{E_1}^{E_2} \quad (71)$$

$$\int_{E_1}^{E_2} \cos E \sin E(1 - e^2 \cos^2 E)^{1/2}\, dE = \frac{1}{3e^2}(1 - e^2 \cos^2 E)^{3/2}\Bigg|_{E_1}^{E_2} \quad (72)$$

$$\int_{E_1}^{E_2} \sin E(1 - e^2 \cos^2 E)^{1/2}\, dE$$

$$= \frac{1}{2e}\left\{\sin^{-1}\left[\sqrt{1 - e^2 \cos^2 E}\right] - e \cos E \sqrt{1 - e^2 \cos^2 E}\right\}\Bigg|_{E_1}^{E_2} \quad (73)$$

$$\int_{E_1}^{E_2} \sin^2 E\,(1 - e^2 \cos^2 E)^{1/2}\, dE$$

$$= \left[\frac{1}{3}\,\frac{(1 - e^2 \cos^2 E)^{1/2} \sin^3 E}{\cos E}\right.$$

$$\left. - \frac{1}{3}\left\{E(\alpha, \phi) + \frac{\cos\phi\sqrt{1 - e^2 \sin^2\phi}}{\sin\phi} - \frac{1}{e^2}\left[(1 - e^2) F(\alpha, \phi) - E(\alpha, \phi)\right]\right\}\right]\Bigg|_{E_1}^{E_2} \quad (74)$$

$$\int_{E_1}^{E_2} \cos^2 E(1 - e\cos E)^{3/2} (1 + e\cos E)^{-1/2}\, dE$$

$$= \frac{2}{e^2}\Bigg[E(\alpha, \phi) - F(\alpha, \phi) + \frac{1}{2}\Big\{e\sin E\sqrt{1 - e^2\cos^2 E}$$

$$+ (1 - e^2)\ln\left[e\sin E + \sqrt{1 - e^2\cos^2 E}\right]\Big\} - \ln\left[e\sin E + \sqrt{1 - e^2\cos^2 E}\right] - \frac{1}{2}\Theta\Bigg]\Bigg|_{E_1}^{E_2} \qquad (75)$$

$$\int_{E_1}^{E_2} \cos E(1 - e\cos E)^{3/2} (1 + e\cos E)^{-1/2}\, dE$$

$$= \frac{1}{2e}\Bigg\{\ln\left[e\sin E + \sqrt{1 - e^2\cos^2 E}\right] - E(\alpha, \phi) + F(\alpha, \phi)$$

$$+ \frac{1}{4}\left[e\sin E\sqrt{1 - e^2\cos^2 E} + (1 - e^2)\ln\left[e\sin E + \sqrt{1 - e^2\cos^2 E}\right]\right]\Bigg\}\Bigg|_{E_1}^{E_2} \qquad (76)$$

$$\int_{E_1}^{E_2} \cos E\sin E(1 - e\cos E)^{3/2} (1 + e\cos E)^{-1/2}\, dE = \frac{1}{e^2}\Bigg\{2\sqrt{1 - e^2\cos^2 E}$$

$$- \frac{1}{3}(1 - e^2\cos^2 E)^{3/2} - e\cos E\sqrt{1 - e^2\cos^2 E} - \sin^{-1}\left[\sqrt{1 - e^2\cos^2 E}\right]\Bigg\}\Bigg|_{E_1}^{E_2} \qquad (77)$$

$$\int_{E_1}^{E_2} \sin E(1 - e\cos E)^{3/2}(1 + e\cos E)^{-1/2}\, dE = \frac{1}{2e}\left\{3\sin^{-1}\left[\sqrt{1 - e^2\cos^2 E}\right]\right.$$

$$\left. - 4\sqrt{1 - e^2\cos^2 E} + e\cos E\sqrt{1 - e^2\cos^2 E}\right\}\Bigg|_{E_1}^{E_2} \quad (78)$$

$$\int_{E_1}^{E_2} \sin^2 E(1 - e\cos E)^{3/2}(1 + e\cos E)^{-1/2}\, dE$$

$$= \frac{1}{e^2}\left\{2\left[(1 - e^2)F(\alpha, \phi) - E(\alpha, \phi)\right]\right.$$

$$+ (1 - e^2)\ln\left[e\sin E + \sqrt{1 - e^2\cos^2 E}\right]$$

$$\left. - e\sin E\sqrt{1 - e^2\cos^2 E} - \Theta\right\}\Bigg|_{E_1}^{E_2} \quad (79)$$

here

$$\Theta = -E(\alpha, \phi) + F(\alpha, \phi) + e^2\left[E(\alpha, \phi) + e^2 F(\alpha, \phi)\right.$$

$$\left. - \frac{\sin\phi\sqrt{1 - e^2\sin^2\phi}}{\cos\phi}\right].$$

Equations (51 through (79) give us the functions G(E) of equation (6.a). For $\rho(E)$, we adopt the expression given by King-Hele (Reference 4),

$$\rho(E) = \rho_p \{1 + b x^2 (1 - \cos E)^2\} e^{-(x/H_p)(1-\cos E)} \tag{80}$$

where ρ_p and H_p are the density and density scale height at perigee respectively, $x = ae$, and $b = \mu/2H_p^2 \left[1 + (1/2)\mu\right]$. Here μ is a constant which represents the rate of increase of H the density scale height, and is most likely to have a value near 0.1. The scale height is defined (Reference 5) as

$$H = \frac{h_p}{1 + R^* M_0 g} \tag{81}$$

where R^* is the universal gas constant, h_p is the pressure scale height given in the Tables, M_0 is the mean molecular weight of air at sea level, and the gravitational constant as a function of altitude is

$$g = \frac{g_0 \ r_e^2}{(r_e + Z)^2} \tag{82}$$

Here r_e is the effective earth's radius at a specific latitude, and Z is the geometric altitude.

Now differentiating equation (80) with respect to the eccentric anomaly, we obtain the following result;

$$d\,\rho\,(E) = \rho_p\, e^{-(x/H_p)(1-\cos E)} \left[\sin E \left(- \frac{x}{H_p} + 2\,b\,x^2 - \frac{b\,x^3}{H_p} \right) - 2\,b\,x^2 \cos E \sin F \left(1 - \frac{x}{H_p} \right) - \frac{b\,x^3}{H_p} \cos^2 E \sin E \right] \tag{83}$$

Equations (51) through (79) and (80) determine the first term of equation (6.a). We now use equation (83) together with (51) through (79) to obtain each integral of equation (6.a). For a fairly circular orbit, the integral term of (6.a) has a small effect since the atmospheric density does not change much.

Multiplying each term of equation (83) by each of the results of equations (51) through (79), expanding each of the radical terms in a Taylor expansion such as,

$$(1 \pm x)^n = 1 \pm n\,x + \frac{n\,(n-1)}{2}\,x^2 \tag{84}$$

and

$$(1 \pm x)^{-n} = 1 \mp n\,x + \frac{n\,(n+1)}{2}\,x^2,$$

Expanding the term,

$$\ln \left(e \sin E + \sqrt{1 - e^2 \cos^2 E}\right)$$

$$= \ln \left[\sqrt{1 - e^2}\right] + \frac{e \sin E}{(1 - e^2)^{1/2}}$$

$$- \frac{1}{(1 - e^2)^{3/2}} \frac{e^3 \sin^3 E}{3!}, \tag{85}$$

and retaining those terms in k^4 (e^4) in equations (49) and (50), we obtain the following set of integrals and solutions,

$$\int_{E_1}^{E_2} E \sin E \, d E = \sin E - E \cos E \Big|_{E_1}^{E_2} \tag{86}$$

$$\int_{E_1}^{E_2} E^2 \sin E \, d E = 2 E \sin E + 2 \cos E - E^2 \cos E \Big|_{E_1}^{E_2} \tag{87}$$

$$\int_{E_1}^{E_2} E^3 \sin E \, d E = (3 E^2 - 6) \sin E - (E^3 - 6 E) \cos E \Big|_{E_1}^{E_2} \tag{88}$$

$$\int_{E_1}^{E_2} E^4 \sin E \, d E = \{- E^4 \cos E + 4 \, [(3 E^2 - 6) \cos E + E^3 \sin E - 6 E \sin E]\} \Big|_{E_1}^{E_2} \tag{89}$$

$$\int_{E_1}^{E_2} E^5 \sin E \, d E = \{- E^5 \cos E + 5 E^4 \sin E - 20 \, I \, (88)\} \Big|_{E_1}^{E_2} \tag{90}$$

where I (88) denotes the result of integral (88) above

$$\int_{E_1}^{E_2} \sin E \cos E \, d E = \frac{1}{2} \sin^2 E - \frac{1}{4} \Big|_{E_1}^{E_2} \tag{91}$$

$$\int_{E_1}^{E_2} E \sin E \cos E \, d E = \frac{1}{2} E \sin^2 E - \frac{E}{4} + \frac{1}{4} \sin E \cos E \Big|_{E_1}^{E_2} \tag{92}$$

$$\int_{E_1}^{E_2} E^2 \sin E \cos E = \frac{1}{4} E \sin 2 E - \frac{1}{4} E^2 \cos 2 E + \frac{1}{8} \cos 2 E \Big|_{E_1}^{E_2} \tag{93}$$

$$\int_{E_1}^{E_2} E^3 \sin E \cos E \, d E = \frac{3}{8} E^2 \sin 2 E - \frac{3}{16} \sin 2 E - \frac{1}{4} E^3 \cos 2 E + \frac{3}{8} E \cos 2 E \Big|_{E_1}^{E_2} \tag{94}$$

$$\int_{E_1}^{E_2} E^4 \sin E \cos E \, d E = \frac{1}{8} \left[(4 E^3 - 6 E) \sin 2 E - (2 E^4 - 6 E^2 + 3) \cos 2 E \right] \Big|_{E_1}^{E_2} \tag{95}$$

$$\int_{E_1}^{E_2} E^5 \sin E \cos E \, d E = \frac{1}{8} \left[(5 E^4 - 15 E^2 + 15 E) \sin 2 E - (2 E^5 - 10 E^3 + 15 E) \cos 2 E \right] \Big|_{E_1}^{E_2} \tag{96}$$

$$\int_{E_1}^{E_2} \sin^2 E \cos E \, d E = \frac{1}{3} \sin^3 E \Big|_{E_1}^{E_2} \tag{97}$$

$$\int_{E_1}^{E_2} \sin E \cos^2 E \, d E = - \frac{1}{3} \cos^3 E \Big|_{E_1}^{E_2} \tag{98}$$

$$\int_{E_1}^{E_2} E^2 \sin^2 E \cos E \, d\, E = \frac{7}{2} E \cos E + \frac{1}{2}\left(\frac{E^2}{2} - 1\right) \sin E$$

$$+ \left(\frac{E^2}{12} - \frac{1}{54}\right) \sin 3\, E + \frac{E}{18} \cos 3\, E \, \Bigg|_{E_1}^{E_2} \tag{99}$$

$$\int_{E_1}^{E_2} \sin^2 E \cos^2 E \, d\, E = \frac{1}{8} E - \frac{1}{32} \sin 4\, E \, \Bigg|_{E_1}^{E_1} \tag{100}$$

$$\int_{E_1}^{E_2} E \sin E \cos^2 E \, d\, E = I\ (86) - \left[\frac{E \cos 3\, E}{12} - \frac{\sin 3\, E}{36}\right.$$

$$\left. - \frac{3\, E \cos E}{4} + \frac{3 \sin E}{4}\right]\Bigg|_{E_1}^{E_2} \tag{101}$$

$$\int_{E_1}^{E_2} E^2 \sin E \cos^2 E \, d\, E = I\ (87) - \left[\left(\frac{1}{12} E^2 + \frac{1}{54}\right) \cos 3\, E\right.$$

$$- \left(\frac{3}{4} E^2 + \frac{3}{2}\right) \cos E + \frac{3}{2} E \sin E \tag{102}$$

$$\left. - \frac{1}{18}\ E \sin 3\, E\right]\Bigg|_{E_1}^{E_2}$$

$$\int_{E_1}^{E_2} E^3 \sin E \cos^2 E \, d\, E = I\ (88) + \left[\frac{1}{12} E^3 \cos 3\, E - \frac{1}{54} E \cos 3\, E\right.$$

$$- \frac{3}{4} E^3 \cos E + \frac{3}{2} E \cos E + \frac{9}{4} E^2 \sin E$$

$$\left.- \frac{1}{12} E^2 \sin 3\, E + \frac{1}{162} \sin 3\, E - \frac{3}{2} \sin E\right]\Bigg|_{E_1}^{E_2} \quad (103)$$

$$\int_{E_1}^{E_2} E^4 \sin E \cos^2 E \, d\, E = E\, I\ (103) - \left\{\frac{21}{4} I\ (87)\right.$$

$$+ \frac{15}{2} \cos E - \frac{7}{4} I\ (88) + \left[\frac{15}{2} (\cos E\right.$$

$$\left.+ E \sin E)\right] + \left[\sin 3\, E \left(\frac{1}{1458} - \frac{2}{81} E\right.\right.$$

$$\left.\left.+ \frac{1}{36} E^3\right)\right] + \left[\cos 3\, E \left(- \frac{2}{243} - \frac{1}{486} E + \frac{1}{36} E^2\right)\right]$$

$$- \left[\frac{1}{72} E^3 + \frac{1}{12} \left(\frac{E^2}{12} - \frac{1}{216}\right) \sin 6\, E\right.$$

$$\left.\left. + \frac{E \cos 6\, E}{432}\right]\right\}\Bigg|_{E_1}^{E_2} \qquad (104)$$

$$\int_{E_1}^{E_2} E^4 \sin^2 E \cos E \, d E = \frac{1}{5} \sin E \cdot I\ (95)$$

$$- \frac{1}{5} I\ (103) + \frac{3}{10} I\ (101) + \frac{12}{5} I\ (99)$$

$$+ \frac{63}{5} \sin E - \frac{6}{5} I\ (97) - \frac{24}{5} E \cos E$$

$$- 6 E^2 \sin E + \frac{2}{5} E^4 \sin E + \frac{2}{5} E^3 \cos E \Bigg|_{E_1}^{E_2} \quad (105)$$

$$\int_{E_1}^{E_2} E^5 \sin E \cos^2 E \, d E = \frac{1}{3} \cos^2 E\ I\ (90)$$

$$+ \frac{10}{3} I\ (105) - 40\ I\ (99) + 80\ I\ (97)$$

$$+ \frac{40}{3} I\ (103) - 40\ I\ (101) \Bigg|_{E_1}^{E_2} \qquad (106)$$

$$\int_{E_1}^{E_2} E^4 \sin E \, d E = \{ - E^4 \cos E + 4\ [(3 E^2 - 6) \cos E$$

$$+ E^3 \sin E - 6 E \sin E] \}_{E_1}^{E_2} \qquad (107)$$

$$\int_{E_1}^{E_2} \sin^2 E \, d E = \frac{E}{2} - \frac{\sin 2 E}{4} \Bigg|_{E_1}^{E_2}$$

$$\int_{E_1}^{E_2} E^2 \sin^2 E \, d E = \frac{E^3}{6} - \left(\frac{E^2}{4} - \frac{1}{8}\right) \sin 2 E - \frac{E \cos 2 E}{4} \Bigg|_{E_1}^{E_2} \tag{109}$$

$$\int_{E_1}^{E_2} E^4 \sin^2 E \, d E = \frac{1}{2} \sin E \, I\,(107) + \frac{E^5}{10} - 2 E^3$$

$$+ 6 \, I\,(109) + 6 E + 3 \sin 2 E$$

$$- 2 \, I\,(94) - 12 \, I\,(92) \Bigg|_{E_1}^{E_2} \tag{110}$$

$$\int_{E_1}^{E_2} \sin^4 E \, d E = \frac{3}{8} E - \frac{1}{4} \sin 2 E + \frac{1}{32} \sin 4 E \Bigg|_{E_1}^{E_2} \tag{111}$$

$$\int_{E_1}^{E_2} \sin E \cos^3 E \, d E = -\frac{1}{4} \cos^4 E \Bigg|_{E_1}^{E_2} \tag{112}$$

$$\int_{E_1}^{E_2} E^2 \sin E \cos^3 E \, d E = \frac{3}{4} \Bigg\{ \cos E \, I\,(102) + \frac{125}{36} \, I\,(91)$$

$$- \frac{2}{27} \, I\,(112) + \frac{1}{12} E^2 - \frac{1}{24} E \sin 2 E - \frac{1}{48} \cos 2 E$$

$$- \frac{2}{9} E \, I\,(111) - \frac{1}{72} \sin^4 E \Bigg\} \Bigg|_{E_1}^{E_2} \tag{113}$$

$$\int_{E_1}^{E_2} E^4 \sin^2 E \cos^2 E \, d E = E^4 \, I \, (100) - \frac{1}{10} E^5$$

$$- \frac{1}{32} E^3 \cos 4 E + \frac{3}{256} E \cos 4 E - \frac{3}{1024} \sin 4 E$$

$$+ \frac{3}{128} E^2 \sin 4 E \Bigg|_{E_1}^{E_2} \tag{114}$$

$$\int_{E_1}^{E_2} \sin E \cos^2 E \sqrt{1 - e^2 \cos^2 E} \, d E = \frac{1}{4 e^2} \Bigg[\frac{e \cos E}{2} \sqrt{1 - e^2 \cos^2 E}$$

$$- e^3 \cos^3 E \sqrt{1 - e^2 \cos^2 E} + \frac{1}{2} \arc \sin \left(\sqrt{1 - e^2 \cos^2 E} \right) \Bigg] \Bigg|_{E_1}^{E_2} \tag{115}$$

$$\int_{E_1}^{E_2} E \sin E \sin 4 E \, d E = \frac{4}{3} E \sin^3 E + \frac{4}{75} \cos E$$

$$- \frac{2}{675} \cos^3 E - \frac{8}{5} E \sin^5 E$$

$$- \frac{8}{25} \sin^4 E \cos E \Bigg|_{E_1}^{E_2} \tag{116}$$

$$\int_{E_1}^{E_2} \sin 4 E \cos E \, d E = - \frac{\cos 5 E}{10} - \frac{\cos 3 E}{6} \Bigg|_{E_1}^{E_2} \tag{117}$$

$$\int_{E_1}^{E_2} E \cos E \, d E = \cos E + E \sin E \Bigg|_{E_1}^{E_2} \tag{118}$$

$$\int_{E_1}^{E_2} E \sin^2 E \cos^3 E \, d E = E \cos E \, I\,(100)$$

$$- \frac{1}{8} I\,(118) + \frac{1}{32} I\,(117) + \frac{1}{8} I\,(87)$$

$$- \frac{1}{32} I\,(116) \Bigg|_{E_1}^{E_2} \tag{119}$$

$$\int_{E_1}^{E_2} E^2 \sin E \cos^4 E \, d E = \frac{\cos^4 E}{5} I\,(87)$$

$$+ \frac{4}{5} \left[2\, I\,(119) - \frac{2 \cos^5 E}{5} \right] \Bigg|_{E_1}^{E_2} \tag{120}$$

$$\int_{E_1}^{E_2} E^2 \sin E \sqrt{1 - e^2 \cos^2 E} \, d E = I\,(87) - \frac{e^2}{2} I\,(102)$$

$$- \frac{e^4}{8} I\,(120) \Bigg|_{E_1}^{E_2} \tag{121}$$

$$\int_{E_1}^{E_2} E \cos^6 E \, d E = \frac{E}{6} \cos^5 E \sin E + \frac{5}{24} E \cos^3 E \sin E$$

$$+ \frac{5\, E^2}{32} + \frac{5}{16} E \sin E \cos E + \frac{1}{36} \cos^6 E$$

$$+ \frac{5}{96} \cos^4 E - \frac{5}{16} I\,(91) \Bigg|_{E_1}^{E_2} \tag{122}$$

$$\int_{E_1}^{E_2} E^2 \sin E \cos^5 E \, d E = - \frac{E^2 \cos^6 E}{6} + \frac{1}{3} I \ (122) \Bigg|_{E_1}^{E_2} \tag{123}$$

$$\int_{E_1}^{E_2} E^2 \sin E \cos E \sqrt{1 - e^2 \cos^2 E} \, d E = I \ (93) - \frac{e^2}{2} I \ (113) - \frac{e^4}{8} I \ (123) \Bigg|_{E_1}^{E_2} \tag{124}$$

$$\int_{E_1}^{E_2} E \cos^7 E \, d E = \frac{1}{7} \cos^6 E \sin E + \frac{6}{35} \cos^4 E \sin E + \frac{4}{15} \cos^2 E \sin E + \frac{8}{15} \sin E \Bigg|_{E_1}^{E_2} \tag{125}$$

$$\int_{E_1}^{E_2} E^2 \sin E \cos^6 E \, d E = - \frac{E^2 \cos^7 E}{7} + \frac{2}{7} I \ (125) \Bigg|_{E_1}^{E_2} \tag{126}$$

$$\int_{E_1}^{E_2} E^2 \sin E \cos^2 E \sqrt{1 - e^2 \cos^2 E} \, d E = I \ (102) - \frac{e^2}{2} I \ (120) - \frac{e^4}{8} I \ (126) \Bigg|_{E_1}^{E_2} \tag{127}$$

$$\int_{E_1}^{E_2} E^3 \cos^5 E \, d\, E = \frac{F^3}{5} \cos^4 E \sin E$$

$$+ \frac{4}{15} E^3 \sin E \cos^2 E + \frac{8}{15} E^3 \sin E$$

$$- \frac{3}{5} I\ (120) - \frac{4}{5} I\ (102)$$

$$- \frac{8}{5} I\ (87) \Bigg|_{E_1}^{E_2} \tag{128}$$

$$\int_{E_1}^{E_2} E^4 \sin E \cos^4 E \, d\, E = - \frac{E^4 \cos^5 E}{5}$$

$$+ \frac{4}{5} I\ (128) \Bigg|_{E_1}^{E_2} \tag{129}$$

$$\int_{E_1}^{E_2} E^4 \sin E \sqrt{1 - e^2 \cos^2 E} \, d\, E = I\ (107)$$

$$- \frac{e^2}{2} I\ (104) - \frac{e^4}{8} I\ (129) \Bigg|_{E_1}^{E_2} \tag{130}$$

$$\int_{E_1}^{E_2} E^4 \sin E \cos^5 E \, d E = -\frac{E^4 \cos^6 E}{6} + \frac{E^3 \sin E \cos^5 E}{9}$$

$$+ \frac{5}{96} E^4 + \frac{5}{24} E^3 \sin E \cos E + \frac{5}{36} E^3 \sin E \cos^3 E$$

$$- \frac{1}{3} I\,(123) - \frac{5}{8} I\,(93) - \frac{5}{12} I\,(113) \Bigg|_{E_1}^{E_2} \tag{131}$$

$$\int_{E_1}^{E_2} E^4 \sin E \cos^3 E \, d E = -\frac{E^4 \cos^4 E}{4} + \frac{3}{32} E^4$$

$$+ \frac{3}{8} E^3 \sin E \cos E + \frac{E^3}{4} \sin E \cos^3 E$$

$$- \frac{9}{8} I\,(93) - \frac{3}{4} I\,(113) \Bigg|_{E_1}^{E_2} \tag{132}$$

$$\int_{E_1}^{E_2} E^4 \sin E \cos E \sqrt{1 - e^2 \cos^2 E} \, d E = I\,(95)$$

$$- \frac{e^2}{2} I\,(132) - \frac{e^4}{8} I\,(131) \Bigg|_{E_1}^{E_2} \tag{133}$$

$$\int_{E_1}^{E_2} E^2 \sin^3 E \, d E = - E^2 \cos E + \frac{1}{3} E^2 \cos^3 E + \frac{3}{2} \cos E$$

$$+ \frac{3}{2} E \sin E - \frac{1}{18} E \sin 3 E - \frac{1}{54} \cos 3 E \Bigg|_{E_1}^{E_2} \tag{134}$$

$$\int_{E_1}^{E_2} E^3 \cos^7 E \, d E = E^3 \left\{\frac{\cos^6 E \sin E}{7} + \frac{6}{35} \cos^4 E \sin E \right.$$

$$\left. + \frac{24}{35} \sin E - \frac{8}{35} \sin^3 E\right\} - \frac{3}{7} I\ (126)$$

$$- \frac{18}{35} I\ (120) - \frac{72}{35} I\ (87) + \frac{24}{35} I\ (134) \Bigg|_{E_2}^{E_2} \qquad (135)$$

$$\int_{E_1}^{E_2} E^4 \sin E \cos^6 E \, d E = - \frac{E^4}{7} \cos^7 E + \frac{4}{7} I\ (135) \Bigg|_{E_1}^{E_2} \qquad (136)$$

$$\int_{E_1}^{E_2} E^4 \sin E \cos^2 E \sqrt{1 - e^2 \cos^2 E} \, d E = I\ (104)$$

$$- \frac{e^2}{2} I\ (129) - \frac{e^4}{8} I\ (136) \Bigg|_{E_1}^{E_2} \qquad (137)$$

$$\int_{E_1}^{E_2} \cos^2 E \sqrt{1 - e^2 \cos^2 E} \, d E = \left(\frac{-(1 - e^2)}{3 e^2} \left\{e^2 \left[\frac{\sin \phi}{\cos \phi} \sqrt{1 - e^2 \sin^2 \phi} \right.\right.\right.$$

$$\left.\left. + \frac{\sin \phi \cos \phi}{\sqrt{1 - e^2 \sin^2 \phi}}\right] - (1 + e^2) \ [\hat{E} - (1 - e^2) \hat{F}]\right\}$$

$$\left. + \frac{1}{3} \left\{\frac{\sin^3 E}{\cos E} \sqrt{1 - e^2 \cos^2 E}\right\}\right) \Bigg|_{E_1}^{E_2} \qquad (138)$$

$$\int_{E_1}^{E_2} \cos^3 E \sqrt{1 - e^2 \cos^2 E}\, d E = \frac{1}{2 e^3} \left\{ \left[1 + \frac{3}{4}(1 - e^2)\right] [e \sin E \sqrt{1 - e^2 \cos^2 E} \right.$$

$$+ (1 - e^2) \ln (e \sin E + \sqrt{1 - e^2 \cos^2 E})]$$

$$\left. - \frac{(1 - e^2 \cos^2 E)^{3/2}\, e \sin E}{2} \right\}_{E_1}^{E_2} \tag{139}$$

$$\int_{E_1}^{E_2} \cos^5 E\, d E = \frac{\cos^4 E \sin E}{5} + \frac{4}{5} \sin E - \frac{4}{15} \sin^3 E \Bigg|_{E_1}^{E_2} \tag{140}$$

$$\int_{E_1}^{E_2} E \sin E \cos^4 E\, d E = - \frac{E \cos^5 E}{5} + \frac{1}{5} I\ (140) \Bigg|_{E_1}^{E_2} \tag{141}$$

$$\int_{E_1}^{E_2} E \sin^3 E\, d E = \frac{E^2}{2} \sin^3 E - \frac{3}{2} I\ (99) \Bigg|_{E_1}^{E_2} \tag{142}$$

$$\int_{E_1}^{E_2} E^2 \cos^5 E\, d E = E^2\, I\ (140) - \frac{8}{5} I\ (86)$$

$$+ \frac{8}{15} I\ (142) \Bigg|_{E_1}^{E_2} \tag{143}$$

$$\int_{E_1}^{E_2} E^2 \cos E\, d E = E^2 \sin E - 2\, I\ (86) \Bigg|_{E_1}^{E_2} \tag{144}$$

$$\int_{E_1}^{E_2} E^2 \cos^3 E \, d\,E = I\ (144) - I\ (99) \Bigg|_{E_1}^{E_2} \tag{145}$$

$$\int_{E_1}^{E_2} E^2 \cos E \sqrt{1 - e^2 \cos^2 E}\, d\,E = I\ (144)$$

$$- \frac{e^2}{2} I\ (145) - \frac{e^4}{8} I\ (143) \Bigg|_{E_1}^{E_2} \tag{146}$$

$$\int_{E_1}^{E_2} E \sin E \cos^5 E \, d\,E = - \frac{E}{6} \cos^6 E + \frac{1}{36} \sin E \cos^5 E$$

$$+ \frac{5}{96} E + \frac{5}{96} \sin E \cos E + \frac{5}{144} \sin E \cos^3 E \Bigg|_{E_1}^{E_2} \tag{147}$$

$$\int_{E_1}^{E_2} \cos^4 E \, d\,E = \frac{3}{8} E + \frac{3}{16} \sin 2\,E$$

$$+ \frac{\cos^3 E \sin E}{4} \Bigg|_{E_1}^{E_2} \tag{148}$$

$$\int_{E_1}^{E_2} E \sin E \cos^3 E \, d\,E = - \frac{E \cos^4 E}{4} + \frac{1}{4} I\ (148) \Bigg|_{E_1}^{E_2} \tag{149}$$

$$\int_{E_1}^{E_2} E^2 \cos^6 E \, d\, E = \frac{E^2}{6} \cos^5 E \sin E + \frac{5}{48} E^3$$

$$+ \frac{5}{16} E^2 \sin E \cos E + \frac{5}{24} E^2 \sin E \cos^3 E$$

$$- \frac{5}{8} I\ (92) - \frac{1}{3} I\ (147) - \frac{5}{12} I\ (149) \Bigg|_{E_1}^{E_2} \tag{150}$$

$$\int_{E_1}^{E_2} E^2 \cos^4 E \, d\, E = \frac{E^2 \cos^3 E \sin E}{4} - \frac{E^3}{4} - \frac{3}{4} I\ (92)$$

$$+ \frac{3}{8} E^2 (E + \sin E \cos E) - \frac{1}{2} I\ (149) \Bigg|_{E_1}^{E_2} \tag{151}$$

$$\int_{E_1}^{E_2} E^2 \cos^2 E \sqrt{1 - e^2 \cos^2 E} \, d\, E = \frac{E^3}{3} - I\ (109)$$

$$- \frac{e^2}{2} I\ (151) - \frac{e^4}{8} I\ (150) \Bigg|_{E_1}^{E_2} \tag{152}$$

$$\int_{E_1}^{E_2} \cos^5 E \, d\, E = \frac{\cos^4 E \sin E}{5} + \frac{4}{5} \sin E$$

$$- \frac{4}{15} \sin^3 E \Bigg|_{E_1}^{E_2} \tag{153}$$

$$\int_{E_1}^{E_2} \cos^7 E \, d\, E = \frac{\cos^6 E \sin E}{7} + \frac{6}{7} I\ (153) \Bigg|_{E_1}^{E_2} \tag{154}$$

$$\int_{E_1}^{E_2} \cos^9 E \, d\, E = \frac{\cos^8 E \sin E}{9} + (8/9)\, I\ (154)\ \Bigg|_{E_1}^{E_2} \tag{155}$$

$$\int_{E_1}^{E_2} E \sin E \cos^6 E \, d\, E = -\frac{E}{7} \cos^7 E + \frac{1}{7}\, I\ (154)\ \Bigg|_{E_1}^{E_2} \tag{156}$$

$$\int_{E_1}^{E_2} \cos^5 E \sqrt{1 - e^2 \cos^6 E}\, d\, E = I\ (153) - \frac{e^2}{2}\, I\ (154)$$

$$- \frac{e^4}{8}\, I\ (155)\ \Bigg|_{E_1}^{E_2} \tag{157}$$

$$\int_{E_1}^{E_2} E^2 \cos^7 E \, d\, E = \frac{E^2}{7} \sin E \cos^6 E + \frac{6}{35} E^2 \sin E \cos^4 E$$

$$+ \frac{24}{35} E^2 \sin E - \frac{8}{35} E^2 \sin^3 E - \frac{12}{35}\, I\ (141)$$

$$- \frac{48}{35}\, I\ (86) + \frac{16}{35}\, I\ (142)$$

$$- \frac{2}{7}\, I\ (156)\ \Bigg|_{E_1}^{E_2} \tag{158}$$

$$\int_{E_1}^{E_2} E^2 \cos^3 E \sqrt{1 - e^2 \cos^2 E}\, d\, E = I\ (145)$$

$$- \frac{e^2}{2}\, I\ (143) - \frac{e^4}{8}\, I\ (158)\ \Bigg|_{E_1}^{E_2} \tag{159}$$

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$$\int_{E_1}^{E_2} E \sin^5 E \, d E = - \frac{E}{5} \sin^4 E \cos E + \frac{4 E}{5} \left[- \cos E + \frac{1}{3} \cos^3 E \right]$$

$$+ \frac{7}{15} \sin E - \frac{1}{9} \sin^3 E + \frac{1}{25} \sin^5 E \Bigg|_{E_1}^{E_2} \tag{160}$$

$$\int_{E_1}^{E_2} E^2 \sin^4 E \cos E \, d E = \frac{E^2}{5} \sin^5 E - \frac{2}{5} I\ (160) \Bigg|_{E_1}^{E_2} \tag{161}$$

$$\int_{E_1}^{E_2} E^3 \sin^3 E \, d E = I\ (88) - I\ (103) \Bigg|_{E_1}^{E_2} \tag{162}$$

$$\int_{E_1}^{E_2} E^3 \sin^3 E \cos^2 E \, d E = \frac{E^3}{5} \sin^4 E \cos E + \frac{E^3}{5} \left[\frac{1}{3} \cos^3 E - \cos E \right]$$

$$- \frac{3}{5} I\ (161) + \frac{3}{5} I\ (144) - \frac{1}{5} I\ (145) \Bigg|_{E_1}^{E_2} \tag{163}$$

$$\int_{E_1}^{E_2} E^4 \sin^2 E \cos^3 E \, d E = \frac{E^4}{5} \sin^3 E \cos^2 E$$

$$+ \frac{2 E^4}{15} \sin^3 E - \frac{4}{5} I\ (153) - \frac{8}{15} I\ (162) \Bigg|_{E_1}^{E_2} \tag{164}$$

$$\int_{E_1}^{E_2} E^4 \cos E \, d E = E^4 \sin E - 4 I\ (88) \Bigg|_{E_1}^{E_2} \tag{165}$$

$$\int_{E_1}^{E_2} E^4 \cos^3 E \, d\, E = E^4 \sin E - 4\, I\,(88) - I\,(105) \Bigg|_{E_1}^{E_2} \tag{166}$$

$$\int_{E_1}^{E_2} E^4 \cos^5 E \, d\, E = I\,(166) - I\,(164) \Bigg|_{E_1}^{E_2} \tag{167}$$

$$\int_{E_1}^{E_2} E^4 \cos E \sqrt{1 - e^2 \cos^2 E} \, d\, E = I\,(165) - \frac{e^2}{2}\, I\,(166)$$

$$- \frac{e^4}{8}\, I\,(167) \Bigg|_{E_1}^{E_2} \tag{168}$$

$$\int_{E_1}^{E_2} E^3 \sin E \cos^3 E \, d\, E = - \frac{E^3 \cos^4 E}{4} + \frac{3}{4}\, I\,(151) \Bigg|_{E_1}^{E_2} \tag{169}$$

$$\int_{E_1}^{E_2} E^4 \cos^4 E \, d\, E = \frac{3\, E^5}{40} + \frac{3\, E^4}{8} \sin E \cos E + \frac{1}{4} E^4 \sin E \cos^3 E$$

$$- \frac{3}{2}\, I\,(94) - I\,(169) \Bigg|_{E_1}^{E_2} \tag{170}$$

$$\int_{E_1}^{E_2} E^4 \cos^2 E \, d\, E = \frac{E^4 \sin E \cos E}{2} + \frac{E^5}{10} - 2\, I\,(94) \Bigg|_{E_1}^{E_2} \tag{171}$$

$$\int_{E_1}^{E_2} E^4 \sqrt{1 - e^2 \cos^2 E} \, d\, E = \frac{E^5}{5} - \frac{e^2}{2}\, I\,(171) - \frac{e^4}{8}\, I\,(170) \Bigg|_{E_1}^{E_2} \tag{172}$$

$$\int_{E_1}^{E_2} E^3 \sin E \cos^5 E \, d E = -\frac{E^2}{6} \cos^6 E + \frac{1}{2} I\ (150) \Big|_{E_1}^{E_2} \tag{173}$$

$$\int_{E_1}^{E_2} E^4 \cos^6 E \, d E = \frac{E^4}{6} \sin E \cos^5 E + \frac{5}{6} E^4\ I\ (148) - \frac{E^5}{4}$$

$$- \frac{2}{3} I\ (173) - \frac{5}{4} I\ (94) - \frac{5}{6} I\ (169) \Big|_{E_1}^{E_2} \tag{174}$$

$$\int_{E_1}^{E_2} E^4 \cos^2 E \sqrt{1 - e^2 \cos^2 E} \, d E = I\ (171)$$

$$- \frac{e^2}{2} I\ (170) - \frac{e^4}{8} I\ (174) \Big|_{E_1}^{E_2} \tag{175}$$

$$\int_{E_1}^{E_2} E^3 \sin E \cos^6 E \, d E = -\frac{E^3}{7} \cos^7 E + \frac{3}{7} I\ (158) \tag{176}$$

$$\int_{E_1}^{E_2} E^3 \sin E \cos^4 E \, d E = I\ (103) - I\ (163) \Big|_{E_1}^{E_2} \tag{177}$$

$$\int_{E_1}^{E_2} E^4 \cos^7 E \, d E = \frac{E^4}{7} \sin E \cos^6 E + \frac{6}{35} E^4 \sin E \cos^4 E$$

$$+ \frac{24\ E^4}{35} \sin E - \frac{8\ E^4}{35} \sin^3 E - \frac{4}{7} I\ (176)$$

$$- \frac{128}{35} I\ (88) - \frac{32}{35} I\ (103) - \frac{24}{35} I\ (177) \Big|_{E_1}^{E_2} \tag{178}$$

$$\int_{E_1}^{E_2} E^4 \cos^3 E \sqrt{1 - e^2 \cos^2 E}\, d\, E = I\ (166)$$

$$- \frac{e^2}{2} I\ (167) - \frac{e^4}{8} I\ (178) \Bigg|_{E_1}^{E_2} \tag{179}$$

$$\int_{E_1}^{E_2} \cos^8 E\, d\, E = \left\{ \frac{\cos^7 E \sin E}{7} + \frac{7}{8} \left[\frac{\cos^5 E \sin E}{6} \right.\right.$$

$$\left.\left. + \frac{5}{6} I\ (148) \right] \right\} \Bigg|_{E_1}^{E_2} \tag{180}$$

$$\int_{E_1}^{E_2} E \sin E \cos^8 E\, d\, E = - \frac{E}{9} \cos^9 E + \frac{1}{9} I\ (155) \Bigg|_{E_1}^{E_2} \tag{181}$$

$$\int_{E_1}^{E_2} \cos^4 E \sqrt{1 - e^2 \cos^2 E}\, d\, E = \left(1 - \frac{5\, e^2}{12}\right) I\ (148)$$

$$- \frac{e^2}{12} \cos^5 E \sin E - \frac{e^4}{8} I\ (180) \Bigg|_{E_1}^{E_2} \tag{182}$$

$$\int_{E_1}^{E_2} E \cos^7 E \sin E\, d\, E = - \frac{E \cos^8 E}{8} + \frac{1}{8} I\ (180) \Bigg|_{E_1}^{E_2} \tag{183}$$

$$\int_{E_1}^{E_2} E^2 \cos^8 E \, d\,E = \frac{E^2}{7} \sin E \cos^7 E + \frac{7}{48} E^2 \sin E \cos^5 E$$

$$+ \frac{35}{192} E^2 \sin E \cos^3 E + \frac{35}{128} E^2 \sin E \cos E$$

$$+ \frac{35}{384} E^3 - \frac{2}{7} I\ (182) - \frac{7}{24} I\ (147)$$

$$- \frac{35}{96} I\ (149) - \frac{35}{64} I\ (92) \Bigg|_{E_1}^{E_2} \tag{184}$$

$$\int_{E_1}^{E_2} E^2 \cos^4 E \sqrt{1 - e^2 \cos^3 E} \, d\,E = I\ (151)$$

$$- \frac{e^2}{2} I\ (150) - \frac{e^4}{8} I\ (183) \Bigg|_{E_1}^{E_2} \tag{185}$$

$$\int_{E_1}^{E_2} E^2 \cos^9 E \, d\,E = \frac{E^2 \sin E \cos^8 E}{9} + \frac{8}{63} E^2 \sin E \cos^6 E$$

$$+ \frac{16}{21} E^2 \left[\frac{\sin E \cos^4 E}{5} + \frac{4}{5} \sin E - \frac{4}{15} \sin^3 E \right] - \frac{2}{9} I\ (181)$$

$$- \frac{16}{63} I\ (156) - \frac{128}{105} I\ (86) - \frac{32}{105} I\ (141)$$

$$- \frac{128}{315} I\ (142) \Bigg|_{E_1}^{E_2} \tag{186}$$

$$\int_{E_1}^{E_2} E^2 \cos^5 E \sqrt{1 - e^2 \cos^2 E}\, d\,E = I\ (143)$$

$$- \frac{e^2}{2} I\ (158) - \frac{e^4}{8} I\ (186) \Bigg|_{E_1}^{E_2} \tag{187}$$

$$\int_{E_1}^{E_2} E^3 \sin E \cos^7 E\, d\,E = - \frac{E^3}{8} \cos^8 E$$

$$+ \frac{3}{8} I\ (184) \Bigg|_{E_1}^{E_2} \tag{188}$$

$$\int_{E_1}^{E_2} E^4 \cos^8 E\, d\,E = \frac{E^4}{8} \sin E \cos^7 E + \frac{7}{48} E^4 \sin E \cos^5 E$$

$$+ \frac{7\,E^5}{128} + \frac{35}{128} E^4 \sin E \cos E + \frac{35}{192} E^4 \sin E \cos^3 E$$

$$- \frac{7}{12} I\ (173) - \frac{35}{32} I\ (94) - \frac{35}{48} I\ (169)$$

$$- \frac{1}{2} I\ (188) \Bigg|_{E_1}^{E_2} \tag{189}$$

$$\int_{E_1}^{E_2} E^4 \cos^4 E \sqrt{1 - e^2 \cos^2 E}\, d\,E = I\ (170)$$

$$- \frac{e^2}{2} I\ (174) - \frac{e^4}{8} I\ (189) \Bigg|_{E_1}^{E_2} \tag{190}$$

$$\int_{E_1}^{E_2} E^3 \sin E \cos^8 E \, d\, E = -\frac{E^3}{9} \cos^9 E + \frac{1}{3} I\ (186) \Bigg|_{E_1}^{E_2} \tag{191}$$

$$\int_{E_1}^{E_2} E^4 \cos^9 E \, d\, E = \frac{E^4}{9} \sin E \cos^8 E + \frac{8\, E^4}{63} \sin E \cos^5 E$$

$$+ \frac{16}{105} E^4 \sin E \cos^4 E + \frac{64}{105} E^4 \sin E$$

$$\frac{-64}{315} E^4 \sin^3 E - \frac{32}{63} I\ (176) - \frac{64}{105} I\ (177)$$

$$- \frac{256}{105} I\ (88) - \frac{256}{315} I\ (162)$$

$$- \frac{4}{9} I\ (191) \Bigg|_{E_1}^{E_2} \tag{192}$$

$$\int_{E_1}^{E_2} E^4 \cos^5 E \sqrt{1 - e^2 \cos^2 E} \, d\, E = I\ (167)$$

$$- \frac{e^2}{2} I\ (178) - \frac{e^4}{8} I\ (192) \Bigg|_{E_1}^{E_2} \tag{193}$$

$$\int_{E_1}^{E_2} E^2 \cos^2 E \, d\, E = \frac{E^3}{6} + \left(\frac{E^2}{4} - \frac{1}{8}\right) \sin 2\, E$$

$$+ \frac{E \cos 2\, E}{4} \Bigg|_{E_1}^{E_2} \tag{194}$$

$$\int_{E_1}^{E_2} E^2 \sqrt{1 - e^2 \cos^2 E} \, d\,E = \frac{E^3}{3} - \frac{e^2}{2} I\ (194)$$

$$- \frac{e^4}{8} I\ (151) \Bigg|_{E_1}^{E_2} \tag{195}$$

$$\int_{E_1}^{E_2} \sin E\, (1 - e^2 \cos^2 E)^{-3/2} \, d\,E = \frac{-\, e \cos E}{\sqrt{1 - e^2 \cos^2 E}} \Bigg|_{E_1}^{E_2} \tag{196}$$

$$\int_{E_1}^{E_2} \sin E \cos E\, (1 - e^2 \cos^2 E)^{-3/2} \, d\,E = \frac{-\,1}{e \sqrt{1 - e^2 \cos^2 E}} \Bigg|_{E_1}^{E_2} \tag{197}$$

$$\int_{E_1}^{E_2} \sin E \cos^2 E\, (1 - e^2 \cos^2 E)^{-3/2} \, d\,E$$

$$= -\frac{1}{e^3} \left\{ \frac{e \cos E}{\sqrt{1 - e^2 \cos^2 E}} + \sin^{-1} \left(\sqrt{1 - e^2 \cos^2 E} \right) \right\} \Bigg|_{E_1}^{E_2} \tag{198}$$

$$\int_{E_1}^{E_2} E^2 \sin E\ \ (1 - e^2 \cos^2 E)^{-3/2} \, d\,E = I\ (87) + \frac{3}{2} e^2\, I\ (102)$$

$$+ \frac{15}{8} I\ (120) \Bigg|_{E_1}^{E_2} \tag{199}$$

$$\int_{E_1}^{E_2} E^2 \sin E \cos E\, (1 - e^2 \cos^2 E)^{-3/2} \, d\,E = I\ (93)$$

$$+ \frac{3}{2} e^2\, I\ (113) + \frac{15}{8}\ e^4\ I\ (123) \Bigg|_{E_1}^{E_2} \tag{200}$$

$$\int_{E_1}^{E_2} E^2 \sin E \cos^2 E \, (1 - e^2 \cos^2 E)^{-3/2} \, d\, E = I \, (102)$$

$$+ \frac{3}{2} e^2 \, I \, (120) + \frac{15}{8} e^4 \, I \, (126) \Bigg|_{E_1}^{E_2} \qquad (201)$$

$$\int_{E_1}^{E_2} E^4 \sin E \, (1 - e^2 \cos^2 E)^{-3/2} \, d\, E = I \, (107)$$

$$+ \frac{3}{2} e^2 \, I \, (104) + \frac{15}{8} e^4 \, I \, (129) \Bigg|_{E_1}^{E_2} \qquad (202)$$

$$\int_{E_1}^{E_2} E^4 \sin E \cos E \, (1 - e^2 \cos^2 E)^{-3/2} \, d\, E = I \, (95)$$

$$+ \frac{3}{2} e^2 \, I \, (132) + \frac{15}{8} e^4 \, I \, (131) \Bigg|_{E_1}^{E_2} \qquad (203)$$

$$\int_{E_1}^{E_2} E^4 \sin E \cos^2 I \, (1 - e^2 \cos^2 E)^{-3/2} \, d\, E = I \, (104) + \frac{3}{2} e^2 \, I \, (129)$$

$$+ \frac{15}{8} e^4 \, I \, (136) \Bigg|_{E_1}^{E_2} \qquad (204)$$

$$\int_{E_1}^{E_2} \sin^3 E \cos E \, (1 - e^2 \cos^2 E)^{-3/2} \, d\, E$$

$$= \frac{1}{e^4} \left[\frac{2 - e^2 \, (1 - \cos^2 E)}{\sqrt{1 - e^2 \cos^2 E}} \right] \Bigg|_{E_1}^{E_2} \qquad (205)$$

$$\int_{E_1}^{E_2} E^2 \sin E \cos^7 E \, d\, E = -\frac{E^2 \cos^8 E}{8} + \frac{1}{4} \left\{ \frac{E \sin E \cos^7 E}{8} \right.$$

$$+ \frac{7\, E \sin E \cos^5 E}{48} + \frac{35\, E \sin E \cos^3 E}{192}$$

$$+ \frac{35\, E \sin E \cos E}{128} + \frac{1}{64} \cos^8 E + \frac{7}{288} \cos^6 E$$

$$\left. + \frac{35}{768} \cos^4 E - \frac{35}{128}\ I\ (91) + \frac{35}{256}\ E^2 \right\} \Bigg|_{E_1}^{E_2} \tag{206}$$

$$\int_{E_1}^{E_2} E^2 \sin^3 E \cos^5 E \, d\, E = I\ (123) - I\ (206) \ \Bigg|_{E_1}^{E_2} \tag{207}$$

$$\int_{E_1}^{E_2} E^2 \sin^3 E \cos E\, (1 - e^2 \cos^2 E)^{-3/2} \, d\, E = I\ (93)$$

$$- I\ (113) + \frac{3}{2} e^2 \left[I\ (113) + I\ (123) \right]$$

$$+ \frac{15}{8} e^4\ I\ (207) \ \Bigg|_{E_1}^{E_2} \tag{208}$$

$$\int_{E_1}^{E_2} E^3 \cos^8 E \, d\,E = \frac{E^3}{8} \sin E \cos^7 E + \frac{7\,E^3 \sin E \cos^5 E}{48}$$

$$+ \frac{35}{512}\, E^4 + \frac{35}{192}\, E^3 \sin E \cos^3 E$$

$$+ \frac{35\,E^3}{128} \sin E \cos E - \frac{3}{8}\, I\,(206)$$

$$- \frac{7}{16}\, I\,(123) - \frac{105}{192}\, I\,(113) - \frac{105}{128}\, I\,(93)\,\Bigg|_{E_1}^{E_2} \qquad (209)$$

$$\int_{E_1}^{E_2} E^4 \sin E \cos^7 E \, d\,E = -\frac{E^4}{8} \cos^8 E + \frac{1}{2}\, I\,(209)\,\Bigg|_{E_1}^{E_2} \qquad (210)$$

$$\int_{E_1}^{E_2} E^4 \sin^3 E \cos E\,(1 - e^2 \cos^2 E)^{-3/2}\, d\,E = \Bigg\{ I\,(95)$$

$$- I\,(132) + \frac{3}{2}\, e^2\, [I\,(132) - I\,(131)]$$

$$+ \frac{15}{8}\, e^4\, [I\,(131) - I\,(210)] \Bigg\}_{E_1}^{E_2} \qquad (211)$$

$$\int_{E_1}^{E_2} \sin E \cos E\,(1 - e^2 \cos^2 E)^{-1/2}\, d\,E = \frac{1}{e^2}\,(1 - e^2 \cos^2 E)^{1/2}\,\Bigg|_{E_1}^{E_2} \qquad (212)$$

$$\int_{E_1}^{E_2} \sin^3 E \cos E \, (1 - e^2 \cos^2 E)^{-1/2} \, d\,E$$

$$= \frac{\sqrt{1 - e^2 \cos^2 E}}{3\, e^4} \left[(1 - e^2 \cos^2 E) - 3\,(1 - e^2)\right] \Bigg|_{E_1}^{E_2} \tag{213}$$

$$\int_{E_1}^{E_2} E^2 \sin E \cos^2 E \, (1 - e^2 \cos^2 E)^{-1/2} \, d\,E = I\,(102)$$

$$+ \frac{e^2}{2}\, I\,(120) + \frac{3}{8}\, e^4\, I\,(156) \Bigg|_{E_1}^{E_2} \tag{214}$$

$$\int_{E_1}^{E_2} E^2 \sin E \cos E \, (1 - e^2 \cos^2 E)^{-1/2} \, d\,E = I\,(93)$$

$$+ \frac{1}{2} e^2\, I\,(113) + \frac{3}{8}\, e^4\, I\,(123) \Bigg|_{E_1}^{E_2} \tag{215}$$

$$\int_{E_1}^{E_2} E^2 \sin^3 E \cos E \, (1 - e^2 \cos^2 E)^{-1/2} \, d\,E = I\,(93)$$

$$+ \left(\frac{e^2}{2} - 1\right)\, I\,(113) + \left(\frac{3}{8}\, e^4 - \frac{e^2}{2}\right) I\,(123)$$

$$- \frac{3}{8}\, e^4\, I\,(206) \Bigg|_{E_1}^{E_2} \tag{216}$$

$$\int_{E_1}^{E_2} E^4 \sin E \cos E \,(1 - e^2 \cos^2 E)\, dE = I\,(95)$$

$$+ \frac{e^2}{2} I\,(132) + \frac{3\,e^4}{8} I\,(131) \Bigg|_{E_1}^{E_2} \qquad (217)$$

$$\int_{E_1}^{E_2} E^4 \sin E \cos^2 E \,(1 - e^2 \cos^2 E)^{-1/2}\, dE = I\,(104)$$

$$+ \frac{e^2}{2} I\,(129) + \frac{3}{8} e^4 I\,(136) \Bigg|_{E_1}^{E_2} \qquad (218)$$

$$\int_{E_1}^{E_2} E^4 \sin^3 E \cos E \,(1 - e^2 \cos^2 E)^{-1/2}\, dE = I\,(95)$$

$$+ \left(\frac{e^2}{2} - 1\right) I\,(132) + \left(\frac{3}{8} e^4 - \frac{e^2}{2}\right) I\,(131)$$

$$- \frac{3}{8} e^4 I\,(210) \Bigg|_{E_1}^{E_2} \qquad (219)$$

$$\int_{E_1}^{E_2} \sin^3 E \cos E \,(1 - e^2 \cos^2 E)^{1/2}$$

$$= \frac{(1 - e^2 \cos^2 E)^{3/2}}{e^4} \left[\frac{(1 - e^2 \cos^2 E)}{5} - \frac{(1 - e^2)}{3}\right] \Bigg|_{E_1}^{E_2} \qquad (220)$$

$$\int_{E_1}^{E_2} \cos^3 E \, dE = \sin E - \frac{1}{3} \sin^3 E \Bigg|_{E_1}^{E_2} \qquad (221)$$

$$\int_{E_1}^{E_2} \cos^6 E \, d\, E = \frac{\sin E \cos^5 E}{6} + \frac{5}{24} \sin E \cos^3 E$$

$$+ \frac{5}{16} \sin E \cos E + \frac{5}{16} E \Bigg|_{E_1}^{E_2} \tag{222}$$

$$\int_{E_1}^{E_2} \sin^2 E \cos^2 E \left(1 - e^2 \cos^2 E\right)^{-3/2} d\, E = \left[\frac{E}{2} + \frac{\sin 2\, E}{4}\right]$$

$$- \frac{3}{2} e^2\, I\ (153) - \frac{15\, e^4}{8}\, I\ (154) + \frac{15\, e^4}{8}\, I\ (222)$$

$$- I\ (221) + \frac{3}{2} e^2\, I\ (148) \Bigg|_{E_1}^{E_2} \tag{223}$$

$$\int_{E_1}^{E_2} E^2 \sin E \left(1 - e^2 \cos^2 E\right)^{-1/2} d\, E = I\ (87)$$

$$+ \frac{e^2}{2}\, I\ (102) + \frac{3}{8} e^4\, I\ (120) \Bigg|_{E_1}^{E_2} \tag{224}$$

$$\int_{E_1}^{E_2} E^4 \sin E \left(1 - e^2 \cos^2 E\right)^{-1/2} d\, F = I\ (107)$$

$$+ \frac{e^2}{2}\, I\ (104) + \frac{3}{8} e^4\, I\ (129) \Bigg|_{E_1}^{E_2} \tag{225}$$

$$\int_{E_1}^{E_2} E^2 \sin^3 \cos E \,(1 - e^2 \cos^2 E)^{1/2}\, d\,E = I\ (93)$$

$$-\left(\frac{1}{2} e^2 + 1\right) \ I\ (113) - \left(\frac{e^4}{8} - \frac{e^2}{2}\right) I\ (123)$$

$$+ \frac{e^4}{8} I\ (206)\ \Bigg|_{E_1}^{E_2} \tag{226}$$

$$\int_{E_1}^{E_2} E^4 \sin^3 E \cos E\, (1 - e^2 \cos^2 E)^{1/2}\, d\,E = I\ (95)$$

$$-\left(\frac{e^2}{2} + 1\right) I\ (132) - \left(\frac{e^4}{8} - \frac{e^2}{2}\right) I\ (131)$$

$$+ \frac{e^4}{8}\ I\ (210)\ \Bigg|_{E_1}^{E_2} \tag{227}$$

$$\int_{E_1}^{E_2} \sin E \cos^5 E\, (1 - e^2 \cos^2 E)^{-3/2}\, d\,E = -\frac{\cos^6 E}{6}$$

$$-\frac{3}{16} e^2 \cos^8 E - \frac{3}{16} e^4 \cos^{10} E\ \Bigg|_{E_1}^{E_2} \tag{228}$$

$$\int_{E_1}^{E_2} E^2 \sin^2 E \cos^2 E \,(1 - e^2 \cos^2 E)^{-3/2}\, d\,E = \frac{E^3}{3} - I\,(109)$$

$$+ \left(\frac{3}{2}\, e^2 - 1\right)\ I\,(151) + \left(\frac{15}{8}\, e^4 - \frac{3}{2}\, e^2\right)\ I\,(150)$$

$$- \frac{15}{8}\, e^4\ I\,(184)\ \Bigg|_{E_1}^{E_2} \tag{229}$$

$$\int_{E_1}^{E_2} E^4 \sin^2 E \cos^2 E \,(1 - e^2 \cos^2 E)^{-3/2}\, d\,E = I\,(171)$$

$$+ \left(\frac{3}{2}\, e^2 - 1\right)\ I\,(170) + \left(\frac{15}{8}\, e^4 - \frac{3}{2}\, e^2\right)\ I\,(174)$$

$$- \frac{15}{8}\, e^4\ I\,(189)\ \Bigg|_{E_1}^{E_2} \tag{230}$$

$$\int_{E_1}^{E_2} E^2 \sin E \cos^9 E\, d\,E = - \frac{E^2}{10} \cos^{10} E + \frac{E^2}{5} \left[\frac{\cos^9 E \sin E}{10}\right.$$

$$+ \frac{9}{10}\left(\frac{\cos^7 E \sin E}{8} + \frac{7}{8}\left(\frac{\cos^5 E \sin E}{6} + \frac{5}{16} E\right.\right.$$

$$\left.\left.\left. + \frac{5}{16} \sin E \cos E + \frac{5}{24} \sin E \cos^3 E\right)\right)\right]$$

$$+ \frac{\cos^{10} E}{500} + \frac{9 \cos^8 E}{3200} + \frac{7 \cos^6 E}{1600}\ \frac{-21\, E^2}{320}$$

$$+ \frac{21}{2560} \cos^4 E\ \frac{-21}{160}\ I\,(91)\ \Bigg|_{E_1}^{E_2} \tag{231}$$

$$\int_{E_1}^{E_2} E^2 \sin E \cos^5 E \,(1 - e^2 \cos^2 E)^{-3/2}\, d\, E = \frac{15}{8}\, e^4\, I\,(231)$$

$$+ I\,(123) + \frac{3}{2} e^2\, I\,(206)\ \Big|_{E_1}^{E_2} \qquad (232)$$

$$\int_{E_1}^{E_2} E^4 \sin E \cos^9 E\, d\, F = -\frac{E^4}{10}\cos^{10} E + \frac{2}{5}\ \left\{\frac{\cos^9 E \sin E}{10}\right.$$

$$+ \frac{9}{10}\left[\frac{\cos^7 E \sin E}{8} + \frac{7}{8}\left(\frac{\cos^5 E \sin E}{6} + \frac{5}{16} E\right.\right.$$

$$\left.\left.\left. + \frac{5}{16} \sin E \cos E + \frac{5}{24} \sin E \cos^3 E\right)\right]\right\}$$

$$- \frac{6}{50}\, I\,(231) - \frac{27}{200}\, I\,(206) - \frac{63}{400}\, I\,(123)$$

$$- \frac{189\, E^2}{1280} - \frac{189}{640}\, I\,(93) - \frac{63}{320}\, I\,(113)\ \Big|_{E_1}^{E_2} \qquad (233)$$

$$\int_{E_1}^{E_2} E^4 \sin E \cos^5 E\,(1 - e^2 \cos^2 E)^{-3/2}\, d\, E = \frac{15}{8}\, e^4\, I\,(233)$$

$$+ I\,(131) + \frac{3}{2} e^2\, I\,(210)\ \Big|_{E_1}^{E_2} \qquad (234)$$

$$\int_{E_1}^{E_2} \frac{\sin^4 (1 - e^2 \cos^2 E)^{1/2}}{\cos E} \, d\, E = -\frac{1}{3} \sin^3 E - \sin E$$

$$- \frac{e^2}{10} \sin^5 E - \frac{e^4}{8} \left[\frac{1}{7} \sin^3 E \left(\frac{2}{5} + \frac{3}{5} \cos^2 E \right.\right.$$

$$\left.\left. - \cos^4 E \right) \right] \Bigg|_{E_1}^{E_2} \tag{235}$$

$$\int_{E_1}^{E_2} \frac{\sin^2 E}{\cos E} (1 - e^2 \cos^2 E)^{1/2} \, d\, E = - \sin E - \frac{1}{2} e^2 \, I \, (97)$$

$$- \frac{e^4}{8} \left[\frac{1}{5} \sin^3 E \left(\frac{2}{3} + \cos^2 E \right) \right] \Bigg|_{E_1}^{E_2} \tag{236}$$

$$\int_{E_1}^{E_2} E \sin^7 E \, d\, E = - \frac{E \sin^4 E \cos E}{7} \left(\sin^2 E + \frac{6}{5} \right)$$

$$+ \frac{8}{35} E \cos E (\cos^2 E - 3) + \frac{1}{7} \left(\frac{1}{7} \sin^7 E \right.$$

$$\left. + \frac{6}{25} \sin^5 E - \frac{8}{15} \sin^3 E + \frac{32}{5} \sin E \right) \Bigg|_{E_1}^{E_2} \tag{237}$$

$$\int_{E_1}^{E_2} E^2 \sin^4 E \cos^3 E \, d\, E = \frac{2}{7} I \, (237) - \frac{E^2}{7} \sin^7 E$$

$$+ I \, (161) \Bigg|_{E_1}^{E_2} \tag{238}$$

$$\int_{E_1}^{E_2} \frac{E^2 \sin^4 E}{\cos E} (1 - e^2 \cos^2 E)^{1/2} \, d\,E = -\frac{e^4}{8} \, I\,(238)$$

$$- \frac{1}{2} e^2 \, I\,(161) - I\,(144) - I\,(99) \Bigg|_{E_1}^{E_2} \tag{239}$$

$$\int_{E_1}^{E_2} E^4 \sin^4 E \cos^3 E \, d\,E = I\,(164) - I\,(167)$$

$$+ I\,(178) \Bigg|_{E_1}^{E_2} \tag{240}$$

$$\int_{E_1}^{E_2} E^4 \sin^4 E \cos E \, d\,E = I\,(105) - I\,(164) \Bigg|_{E_1}^{E_2} \tag{241}$$

$$\int_{E_1}^{E_2} \frac{E^4 \sin^4 E}{\cos E} \, d\,E = I\,(165) - I\,(105) \Bigg|_{E_1}^{E_2} \tag{242}$$

$$\int_{E_1}^{E_2} \frac{E^2 \sin^4 E \, (1 - e^2 \cos^2 E)^{1/2}}{\cos E} \, d\,E = I\,(242)$$

$$- \frac{e^2}{2} \, I\,(241) - \frac{e^4}{8} \, I\,(240) \Bigg|_{E_1}^{E_2} \tag{243}$$

$$\int_{E_1}^{E_2} \frac{E^2 \sin^2 E \, (1 - e^2 \cos^2 E)^{1/2}}{\cos E} \, dE = - \frac{e^2}{2} I\,(99)$$

$$- I\,(144) - \frac{e^4}{8} I\,(119) \Bigg|_{E_1}^{E_2} \tag{244}$$

$$\int_{E_1}^{E_2} \frac{E^4 \sin^2 E \, (1 - e^2 \cos^2 E)^{1/2}}{\cos E} = - \frac{e^2}{2} I\,(105)$$

$$- \frac{e^4}{8} I\,(164) - I\,(165) \Bigg|_{E_1}^{E_2} \tag{245}$$

Thus, both terms of equation (6.a) are known for each of the Vinti orbital elements, and the air-drag effect for any eccentricity is also known.

REMARKS

The derivation of the drag rate equations is perfectly general and stems only from first principles. That is, since we employ an angular momentum argument, these equations represent a time deviation of the orbital elements from forces conservative or nonconservative. Here we have solved them from semiempirically determined force functions of atmospheric drag through the eccentric anomaly. In all of our test cases, we used only a static model atmosphere. Next is the inclusion of a more refined day-night profile.

The integrated rate equations now permit one to include drag by reinitialization; after solving Vinti's kinetic equations of motion for a given value of time,

the computed orbital parameters are then used to obtain the solutions of the rate equations for that same time. The latter results are then used to obtain the new constants of the motion in Vinti's theory which arise in the factorization of certain quartic polynomials, necessary in the separation of the Hamilton-Jacobi equation. This entire process is carried out for each given time. It is confirmed that for satellites encountering sufficient aerodynamic drag, this procedure adds considerable accuracy in the drag region, and allows one to perform fewer corrections on the orbit. It must be emphasized, that this procedure is entirely different from that of numerical integration. In the Vinti program with aerodynamic drag, the rate equations serve only to provide new initial conditions as functions of time through the orbital elements, with which separability can be achieved, and the kinetic equations of motion can be solved (Reference 2).

A Fortran version of this program has been prepared and tested on the IBM 360/95 electronic digital computer. At present, the Fortran program does not include the second term of equation (6.a), (Equations (86) through (245)) which account for drag variations on the satellite from changing atmospheric density. This part is presently being added to the Vinti orbit determination program. On the other hand, if the eccentricity is not too high, the second term of equation (6.a) may not have a great effect, providing perigee is not low also. In any event, it is interesting to observe the effect of even the first term in equation (6.a) on the accuracy of a heavy air-drag satellite. Of concern are the San Marco 2 and S-55 Micrometeor Impact Satellites. To date, only the San Marco 2

satellite has been tested. The initial orbital data for this satellite are as follows:

Semimajor axis	6807.0374 km
Eccentricity	0.03333142
Perigee	201.7898 km
Apogee	655.5663 km
Inclination	2.89095642 degrees
Period	93.15 minutes

Figure 1 is a plot of the time variation of the Izsak elements after reinitializing the Vinti program using these initial conditions. Their behavior is exactly what is expected (see Sterne, Reference 1). Figures 2 and 3 are the size of residuals of computed and observed direction cosine differences ΔL and ΔM versus time. The comparison was for a three day period. In both cases the non-drag and drag orbits were comparable for approximately twenty hours. After this, due to heavy air drag, the difference was large, as much as an order of magnitude. In the drag calculations, only a static atmospheric model was used. However, this is being replaced by a more refined day-night profile model with which somewhat more accuracy should be obtained. At any rate, the results are striking, and it is felt that with the addition of equations (86) through (245) (the second term of equation (6.a)), there exists an analytic drag-gravitational theory or program operating as a fundamental unit to account for the motion of an artificial Earth satellite.

The drag program has also been incorporated into the differential correction portion of the Vinti program. This has added the following advantages: Since

from Figures 2 and 3 it is seen that the Vinti orbit generator can predict approximately three and one-half times longer with the drag added, an orbit correction to obtain mean elements can be made using the seventy-two hour arc instead of the twenty hour arc. Secondly, the mean fitted elements now reflect or contain the characteristics of atmospheric perturbations as well as those of the full Vinti gravitational potential. There results a more accurate prediction over a much longer period of time, without the need of frequent orbit corrections. This program is presently being prepared for testing the S-55 Micrometeor Impact satellite.

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REFERENCES

1. Sterne, Theodore E., "An Introduction to Celestial Mechanics," Interscience Publishers, Inc., New York, 1960.

2. Vinti, John P., "Improvement of the Spheroidal Method for Artificial Satellites," Astronomical Journal, Volume 74, Number 1, pages 25-34, February 1969.

3. Zur Capellen, W. Meyer, "Integraltafeln," Springer-Verlag, Berlin/Götingen/ Heidelberg, 1960.

4. King-Hele, Desmond, "Theory of Satellite Orbits in an Atmosphere," Butterworth and Co. (Publishers) LTD., London, 1964.

5. U. S. Standard Atmosphere Supplements, 1966.

APPENDIX

As mentioned in Section II above, we shall give here some examples on the solution of the types of integrals encountered there. Basically, all of the different integrals are approached in the same manner as we shall show here.

Given the integral,

$$\int_{E_1}^{E_2} (1 - e^2 \cos^2 E)^{1/2}\, dE$$

Let

$$x^2 = 1 - e^2 \cos^2 E$$

$$dE = \frac{x\, dx}{e^2 \cos E \sin E}$$

$$\cos^2 E = \frac{1 - x^2}{e^2}$$

$$\sin^2 E = \frac{e^2 - 1 + x^2}{e^2}$$

let

$$k^2 = 1 - e^2$$

$$\sin^2 E = \frac{x^2 - k^2}{e^2}$$

$$\int_{E_1}^{E_2} (1 - e^2 \cos^2 E)^{1/2}\, dE = \int_{E_1}^{E_2} \frac{x^2\, dx}{(1 - x^2)^{1/2} (x^2 - k^2)} = - E(\alpha, \phi)\Big|_{E_1}^{E_2}$$

from Reference 3. For the integral,

$$\int_{E_1}^{E_2} \sin^3 E \cos E (1 + e \cos E)^{1/2} (1 - e \cos E)^{-3/2}\, dE$$

$$= \int_{E_1}^{E_2} \frac{\sin^3 E \cos E (1 + e \cos E)^{1/2}}{(1 - e \cos E)(1 - e \cos E)^{1/2}}\, dE$$

multiply numerator and denominator by $(1 - e \cos E)^{1/2}$

$$\int_{E_1}^{E_2} \frac{\sin^3 E \cos E (1 + e \cos E)^{1/2} (1 - e \cos E)^{1/2}}{(1 - e \cos E)(1 - e \cos E)^{1/2} (1 - e \cos E)^{1/2}}\, dE$$

$$= \int_{E_1}^{E_2} \frac{\sin^3 E \cos E (1 - e^2 \cos^2 E)^{1/2}}{(1 - e \cos E)^2}\, dE$$

Now use the same substitutions of example 1 ($x^2 = 1 - e^2 \cos^2 E$)

$$\int_{E_1}^{E_2} \frac{\sin^3 E \cos E\, x^2\, dx}{e^2 \sin E \cos E \left[1 - (1 - x^2)^{1/2}\right]^2}$$

$$= \frac{1}{e^2} \int_{E_1}^{E_2} \frac{\sin^2 E\, x^2\, dx}{\left[1 - (1 - x^2)^{1/2}\right]^2}$$

$$= \frac{1}{e^2} \int_{E_1}^{E_2} \frac{\frac{x^2 - k^2}{e^2} \cdot x^2\, dx}{\left[1 - (1 - x^2)^{1/2}\right]^2}$$

$$= \frac{1}{e^4} \int_{E_1}^{E_2} \frac{(x^2 - k^2)\, x^2}{\left[1 - (1 - x^2)^{1/2}\right]^2}\, dx$$

$$= \frac{1}{e^4} \int_{E_1}^{E_2} \frac{x^4\, dx}{\left[1 - (1 - x^2)^{1/2}\right]^2} - \frac{k^2}{e^4} \int_{E_1}^{E_2} \frac{x^2\, dx}{\left[1 - (1 - x^2)^{1/2}\right]^2}$$

multiply numerator and denominator by

$$\left[1 + (1 - x^2)^{1/2}\right]^2$$

$$\frac{1}{e^4}\int_{E_1}^{E_2} \frac{x^4\left[1+(1-x^2)^{1/2}\right]^2 dx}{\left[1-(1-x^2)^{1/2}\right]^2\left[1+(1-x^2)^{1/2}\right]^2}$$

$$-\frac{k^2}{e^4}\int_{E_1}^{E_2} \frac{x^2\left[1+(1-x^2)^{1/2}\right]^2 dx}{\left[1-(1-x^2)^{1/2}\right]^2\left[1+(1-x^2)^{1/2}\right]^2}.$$

Now use the fact that $(a-b)^2(a+b)^2 = (a^2-b^2)^2$

$$= \frac{1}{e^4}\int_{E_1}^{E_2} \frac{x^4\left[1+(1-x^2)^{1/2}\right]^2 dx}{\left[1-(1-x^2)\right]^2}$$

$$-\frac{k^2}{e^4}\int_{E_1}^{E_2} \frac{x^2\left[1+(1-x^2)^{1/2}\right]^2 dx}{\left[1-(1-x^2)\right]^2}$$

$$= \frac{1}{e^4}\int_{E_1}^{E_2} \frac{x^4\left[1+2(1-x^2)^{1/2}+(1-x^2)\right] dx}{x^4}$$

$$-\frac{k^2}{e^4}\int_{E_1}^{E_2} \frac{x^2\left[1+2(1-x^2)^{1/2}+(1-x^2)\right] dx}{x^4}$$

$$= \frac{1}{e^4}\int_{E_1}^{E_2} \left[2+2(1-x^2)^{1/2}-x^2\right] dx$$

$$-\frac{k^2}{e^4}\int_{E_1}^{E_2} \frac{1}{x^2}\left[2+2(1-x^2)^{1/2}-x^2\right] dx$$

$$= \frac{1}{e^4}\left\{\int_{E_1}^{E_2} 2\,dx + \int_{E_1}^{E_2} 2(1-x^2)^{1/2}\,dx\right.$$

$$- \int_{E_1}^{E_2} x^2\,dx - k^2 \int_{E_1}^{E_2} \frac{2}{x^2}\,dx$$

$$\left. - k^2 \int_{E_1}^{E_2} \frac{2(1-x^2)^{1/2}}{x^2}\,dx + k^2 \int_{E_1}^{E_2} dx\right\}$$

$$= \frac{1}{e^4}\left\{(2+k^2)\int_{E_1}^{E_2} dx + 2\int_{E_1}^{E_2} (1-x^2)^{1/2}\,dx\right.$$

$$- \int_{E_1}^{E_2} x^2\,dx - 2k^2 \int_{E_1}^{E_2} \frac{dx}{x^2}$$

$$\left. - 2k^2 \int_{E_1}^{E_2} \frac{(1-x^2)^{1/2}\,dx}{x^2}\right\}$$

$$= \frac{1}{e^4}\left\{(2+k^2)\,x + 2\cdot\frac{1}{2}\left[x(1-x^2)^{1/2} + \sin^{-1} x\right] - \frac{x^3}{3} + 2k^2\,\frac{1}{x}\right.$$

$$\left. + 2k^2 \left[\frac{(1-x^2)^{1/2}}{x} + \sin^{-1} x\right]\right\}\Bigg|_{E_1}^{E_2}$$

$$= \frac{1}{e^4}\left\{(2+k^2)\,x + x(1-x^2)^{1/2} + \sin^{-1} x \right.$$

$$\left. - \frac{x^3}{3} + \frac{2k}{x} + \frac{2k^2(1-x^2)^{1/2}}{x} + 2k^2 \sin^{-1} x\right\}\Bigg|_{E_1}^{E_2} .$$

Now substitute $x^2 = 1 - e^2\cos^2 E$ back in for x

$$= \frac{1}{e^4}\left\{(1-e^2\cos^2 E)^{1/2}\,(3 - e^2 + e\cos E)\right.$$

$$+ (3-2e^2)\sin^{-1}\left[(1-e^2\cos^2 E)^{1/2}\right]$$

$$- \frac{1}{3}(1-e^2\cos^2 E)^{3/2} + 2(1-e^2)$$

$$\left. \times (1-e^2\cos^2 E)^{-1/2}\,(1+e\cos E)\right\}\Bigg|_{E_1}^{E_2}$$

$$= \int_{E_1}^{E_2} \sin^3 E\cos E(1+e\cos E)^{1/2}\,(1-e\cos E)^{-3/2}\,dE \quad .$$

Finally, for

$$\int_{E_1}^{E_2} e\cos E(1-e^2\cos^2 E)^{1/2}\,dE$$

use the same substitution as used in first two examples. Then this becomes

$$\int_{E_1}^{E_2} \frac{e \cos E x^2\, dx}{e^2 \cos E \sin E} = \int_{E_1}^{E_2} \frac{x^2\, dx}{\frac{e(x^2 - k^2)^{1/2}}{e}} = \int_{E_1}^{E_2} \frac{x^2\, dx}{(x^2 - k^2)^{1/2}} \,.$$

Now let

$$x = k \sec t$$

$$dx = k \sec t \tan t\, dt$$

$$= \int_{E_1}^{E_2} \frac{k^2 \sec^2 t\, k \sec t \tan t\, dt}{\left[k^2 \sec^2 t - k^2\right]^{1/2}}$$

$$= \int_{E_1}^{E_2} \frac{k^3 \sec^3 t \tan t\, dt}{k\left[\sec^2 t - 1\right]^{1/2}}$$

$$= \int_{E_1}^{E_2} \frac{k^3 \sec^3 t \tan t\, dt}{k(\tan^2 t)^{1/2}}$$

$$= \int_{E_1}^{E_2} k^2 \sec^3 t\, dt$$

$$= k^2 \left\{\frac{1}{2} \left[\sec t \tan t + \ln(\sec t + \tan t)\right]\right\} \Bigg|_{E_1}^{E_2}$$

$$\sec t = \frac{x}{k}, \quad \tan t = (\sec^2 t - 1)^{1/2} = \left(\frac{x^2}{k^2} - 1\right)^{1/2} = \frac{1}{k}(x^2 - k^2)^{1/2}$$

$$k^2 \left\{\frac{1}{2}\left[\sec t \tan t + \ln(\sec t + \tan t)\right]\right\}\Bigg|_{E_1}^{E_2}$$

$$= \frac{1}{2} k^2 \left\{\frac{x}{k^2}(x^2 - k^2)^{1/2}\right.$$

$$\left. + \ln\left[\frac{x + (x^2 - k^2)^{1/2}}{k}\right]\right\}\Bigg|_{E_1}^{E_2}$$

$$= \frac{1}{2}\left\{x(x^2 - k^2)^{1/2}\right.$$

$$\left. + k^2 \ln\left[\frac{x + (x^2 - k^2)^{1/2}}{k}\right]\right\}\Bigg|_{E_1}^{E_2}$$

Substitute back in for

$$x^2 = 1 - e^2 \cos^2 E$$

$$= \frac{1}{2}\left\{e \sin E(1 - e^2 \cos^2 E)^{1/2}\right.$$

$$\left. + (1 - e^2) \ln\left[\frac{e \sin E + (1 - e^2 \cos^2 E)^{1/2}}{(1 - e^2)^{1/2}}\right]\right\}\Bigg|_{E_1}^{E_2}$$

$$= \int_{E_1}^{E_2} e \cos E(1 - e^2 \cos^2 E)^{1/2}\, dE \,.$$

These three examples give the general methods used in solving the integrals.

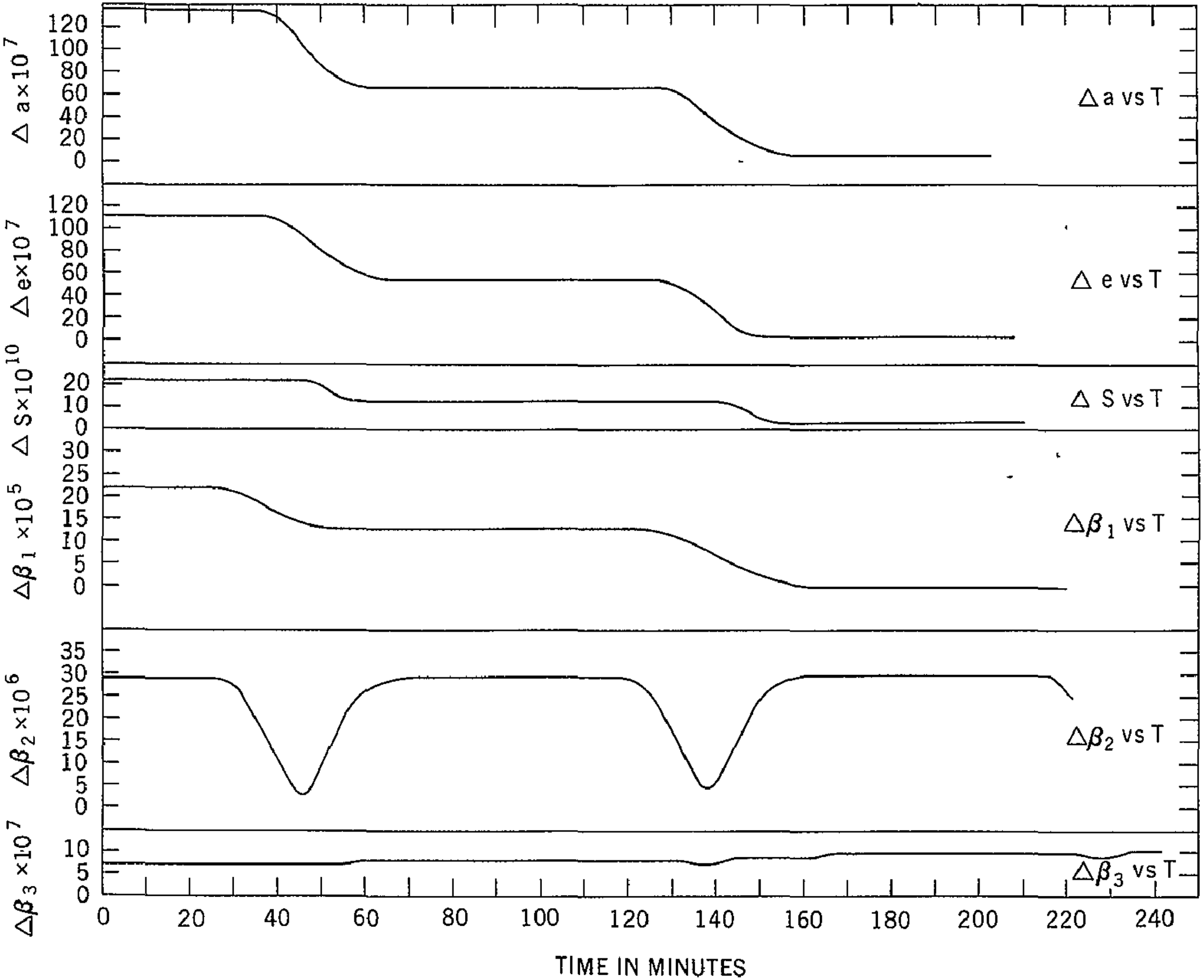

Figure 1. San Marco 2 Satellite. Variation of Elements

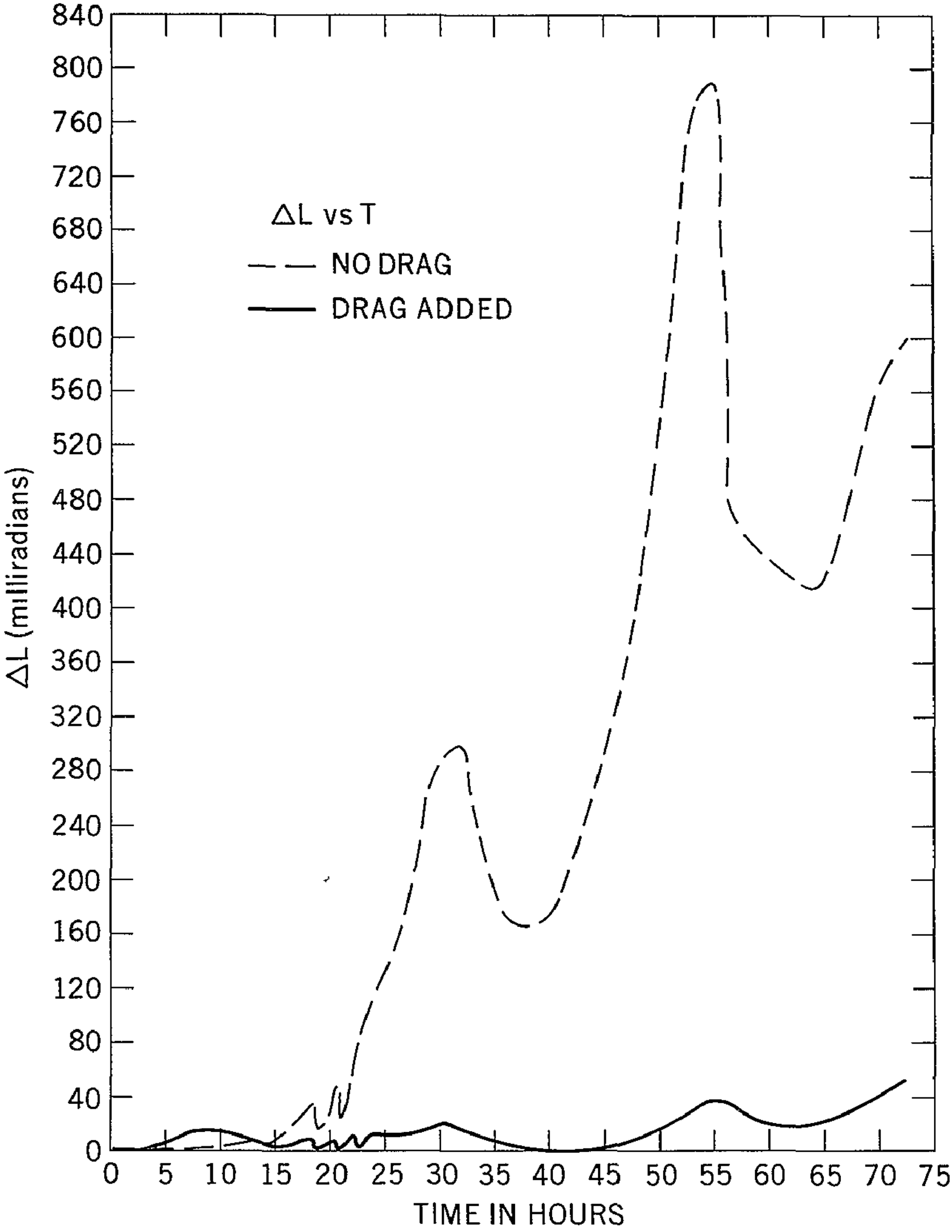

Figure 2. San Marco 2 Satellite (ΔL vs. T)

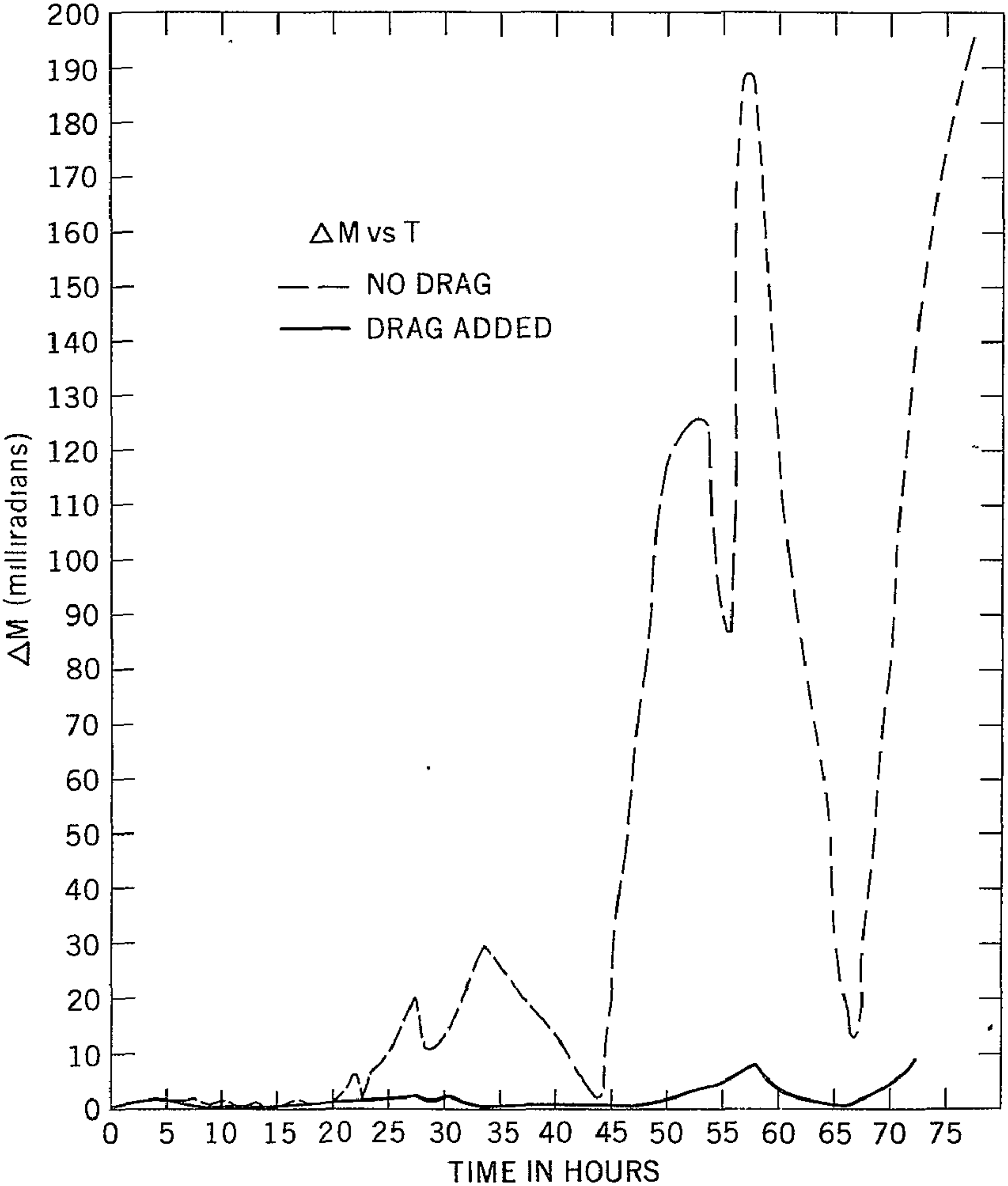

Figure 3. San Marco 2 Satellite (ΔM vs. T)